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ABSTRACT
This paper considers situations in which a set of agents has to decide whether to carry out a given public project or its alternative when agents hold private information. I propose the use of the individually-rational and budget-balanced multibidding mechanism according to which the game to be played by participants has only one stage and simple rules as defined by Pérez-Castrillo and Wettstein (2002) under complete information. It can be applied in a wide range of situations, and its symmetric Bayes-Nash equilibria deliver ex post efficient outcomes if the number of players is two - for any underlying symmetric distribution characterizing uncertainty - or very large.

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1 Introduction

The presence of external effects and/or public goods in an economy makes market mechanisms unreliable for allocating resources efficiently. Inefficiency appears in the form of unexploited gains that can be eliminated by side payments and rearrangements in the distribution of goods. However, it is usually unclear which mechanism to use for implementing the suggested improvements. In the present paper, I study situations in which externalities and/or public goods exist and the members of the society hold important private information related to the problem that is undisclosed to the others. I propose a concrete mechanism for the family of problems and argue that with it, under some conditions, efficiency as social goal can be achieved. Let us first see an example of the type of situations that form this family.

Imagine that a noxious recycling center has to be built according to some political plans. The feasibility studies have already identified two potential areas that are suitable for hosting the site. The decision to be made by the government is to choose between these two areas (projects) trying to take into account its implications on social welfare. In particular, the government's goal is to locate the recycling center where its aggregate positive(/negative) impact is the highest(/lowest). Supposing that parties hold private information (private valuations) on the effects of the recycling center, it is in the best interest of the authority to find out as much as possible about individual private valuations. In order to do so, it can force the affected parties to take part in a *procedure* or mechanism that may make reduce the informational asymmetries.

As for the impact of the site on its surroundings, one can consider the following two scenarios: In the first, the recycling center only affects people in its immediate area, i.e. in the settlement that is located closest to it. This reduces the number of interested parties in the problem to two (plus the central government whose unique objective is to reach a socially efficient decision) and causes positive or negatives changes in the welfare of at most two parties. In the second scenario, the recycling center not only affects the population in its host town, but a larger set of people at the same time as it might influence social welfare across state and country borders. Because of the different nature of the problem, the cases in which there are two and more than two parties will be discussed separately.

Problems of the type described above have already been analyzed in the literature. Under complete information, when parties have precise information on how the others value the projects, the multibidding game proposed and studied by Pérez-Castrillo and Wettstein (2002) can be used efficiently. Without formal definitions, in cases of choices between two projects this mechanism operates as follows:

- Strategies: each participant (each of the affected parties) announces two bids, one for each of the available projects such that these bids sum up to zero.

- Outcomes: the planner sums the bids for every project and chooses the project with the highest aggregate bid as the winner. In case of a tie, some device is used to choose the winner among the projects with the highest aggregate bid. The winning project is carried out, the bids related to it are paid and the surplus (the aggregated bid) is shared among all the agents in equal parts.

Note that the mechanism has a unique (bidding) stage and each agent is asked to bid for all the available projects. Besides each agent is forced to pay her bid given for the project that has been chosen winner. Since the revenue raised by the bidding is given back entirely to participants in equal shares, the multibidding game is budget-balanced. In the complete-information setting, Pérez-Castrillo and Wettstein (2002) showed that in every Nash equilibrium of the bidding, the winning project is efficient. Also any Nash equilibrium of the multibidding mechanism is also a strong Nash equilibrium. For its appealing properties under complete information, its simplicity and feasibility in a wide range of problems, I propose the use of the multibidding mechanism under uncertainty, i.e. incomplete information.

In this paper, I study how the *multibidding mechanism* performs when agents hold private information and are uninformed about others' preferences. I consider ex ante identical risk neutral players and a continuum of possible private valuations; i.e. the continuous case, and study the theoretical properties of the multibidding mechanism with two alternatives. By its definition, the mechanism is safe both to run and to participate, because it is budget-balanced and individually rational once supposed that agents can not escape from the effects of the chosen public project.

In the multibidding game, bids must sum up to zero for every participant. This feature aims at extracting individual private information on the relative valuations between the projects. The mechanism succeeds in it, as at the symmetric Bayes-Nash equilibria participants' bids depend on the difference between private valuations for the alternatives. The equilibrium bidding function is strictly increasing and continuous. Its curvature is determined by the underlying uncertainty that also involves the number of agents.

I show that under uncertainty the multibidding mechanism is always efficient in the two-player, two-project case if the prior distributions are symmetric or players are antagonistically asymmetric.[1] Efficiency is tied to additional conditions when there are more players. The number of agents must be large or with a similar intuition behind uncertainty must be large with zero expected value, in order to achieve efficient outcomes.

[1]Symmetry of distibution means symmetry of the density function around zero. Asymmetry of players refers to situations in which players tend to prefer different projects and form prior beliefs in the opposite way; i.e., player 1 is identical to player 2 with switched project labels.

The two-player, two-project case has been widely analyzed in the auction literature. McAfee (1992) studies simple mechanisms, explores their properties under uncertainty, and presents results for an environment with constant absolute risk aversion. He finds that the winner's bid auction reaches (allocative) efficiency in the chosen set-up. As for the multibidding mechanism, it is important to point out that private valuations are now attached to projects as well as the object in question. This feature makes the model a slightly more general even in the two-agent case. Normally, both parties are eager to win the object and feel bad if it is their opponent who does so. Normalization of payoffs can get us back to the situation studied in the auction literature where players receive zero pay-off when not winning the auction. There also exist problems in which the object is bad and both wish that the other one will get it. These situations can be efficiently dealt with using, for example, a first-price sealed-bid auction with the proper definition for *bids* and *winner*. However, one might imagine situations in which agents share the same opinion and, for instance, both wish that agent 1 get the object. Under these circumstances, the first-price sealed-bid auction is a feasible mechanism once we generalize it, allowing for both negative and positive bids. The multibidding mechanism can also be used without modification in this case.

An important part of the environments considered here has been studied in the literature that deals with the problem of siting noxious facilities. Several sealed-bid mechanisms have been proposed for the problem. The first to suggest the use of an auction in this situation were Kunreuther and Kleindorfer (1986). They showed that outcomes realized by min-max strategies in a low-bid auction are efficient as long as the non-hosting participants are indifferent between all outcomes. For the case of two cities, O'Sullivan (1993) proved that symmetric Bayes-Nash equilibria of the modified low bid game yield an efficient outcome when private valuations are independently drawn.[2] He argues that min-max strategies deliver problematic equilibria in which beliefs may be inconsistent. The rationality of participation, however, is conditional on the compensation for the host city.

Ingberman (1995) analyzed the siting problem with costs depending on the distance from the noxious site and using a majority vote approach. He concluded that decisions reached in this manner would be inefficient, as markets would produce an excessive number of noxious facilities and place them in the wrong sites. Rob (1989) modelled the problem between a pollution-generating firm and the residents as a mechanism design approach for the siting problem. Notice that my model is different in that I assume that the planner is uninterested in revenue-raising. In Rob (1989), binary decision must be made, accept or

[2]It is a voluntary auction under which the city submitting the low bid hosts the region's noxious facility and receives the high bid as compensation.

reject the construction of a pollution-generating plant, and compensatory payments need to be determined. The outcomes of the resulting mechanism are sometimes inefficient. In contrast to the equilibrium outcomes of the multibidding mechanism inefficiencies become rampant when there are many residents affected by pollution and the degree of uncertainty is large.

Jehiel et al. (1996) analyzed a similar model in which external effects appear as the value of a project to an agent depends on the identity of who carries it out. Their set-up includes a seller who wants to sell an object to one of n agents and they characterize the individually rational and incentive compatible mechanisms that maximize the seller's revenue. Revenue maximization is not in my interest in this paper and there are other important assumptions that I do not make. For example, in Jehiel et al. (1996) agents not only know their own valuation, but also the externality they impose on other players.

The well-known Vickrey-Clarke-Groves mechanisms are designed for similar problems, to choose a public project to carry out, under uncertainty and for them truthtelling is a dominant strategy. Therefore, these mechanisms result in efficient outcomes, however they are not budget-balanced. The surplus generated by payments is a loss for the agents.

D'Aspremont and Gérard-Varet (1979) proposed a mechanism that works in a public good set-up under uncertainty with independent types. That mechanism works similarly to the Vickrey-Clarke-Groves schemes, but it substitutes dominant-strategy incentive-compatibility with Bayesian incentive-compatibility. This helps to overcome budget-balance problems and still ex post efficiency is guaranteed. However, a problem still exists: voluntary participation or individual rationality cannot be reached with the proposed mechanism in their set-up.[3]

The rest of this paper is organized as follows. The next section introduces the mechanism formally and starts studying its theoretical properties with symmetric underlying distributions modelling uncertainty. The analysis is done separately in different sections for the two-player and n-player case because of the differences in the techniques and results. I comment on the consequences of asymmetric distributions in Section 5, and relate the multibidding game to a special problem that frequently arises in the literature: a dissolving partnership. Section 6 concludes. Proofs are presented in the appendix.

2 Multibidding game under uncertainty

Consider a set of alternatives $P = \{1, 2\}$ and a set of risk neutral agents $N = \{1, \ldots, i, \ldots, n\}$ whose utility depends on the alternative carried out. I shall denote by $x_i^j \in X \subset \mathbb{R}$ the

[3]A more detailed review on the topic including the Vickrey-Clarke-Groves mechanisms can be found in Jackson (2001).

utility that player i enjoys when project j is the winning project. These values are private information and will be treated as random draws from some underlying common distribution with density $f_{x^j}(x)$ and cumulative distribution function $F_{x^j}(x)$. Agents are identical ex ante; i.e., these functions do not vary across agents, but may do so across projects. I also make the usual assumption of these being common knowledge. The variables x_i^j are considered as continuous random variables here, though my results apply also in the discrete case with the proper adaptation of the concepts to the discrete environment.

A mechanism is called *ex post efficient* if it picks out efficient projects for every possible private valuation profile. Project j is (ex post Pareto) efficient if $\sum_{i \in N} x_i^j \geq \sum_{i \in N} x_i^k$ for all $k \in P$. With this, the social planner's objective is identified.

The *multibidding mechanism* can be formally defined as follows:

In the unique stage of the game, agents simultaneously submit a vector of two real numbers, one for each available project, that sum up to zero. These numbers are called bids where B_i^j denotes agent i's bid for project j.

The project with the highest aggregated bid $B_N^j = \sum_{i \in N} B_i^j$ is chosen winner. Ties are broken randomly.

Once chosen, the winning project is carried out and agents enjoy the utility that it delivers. They also must pay/receive their bids submitted for the winning project and they are returned the aggregated winning bid in equal shares. For example, if project j has obtained the largest aggregated bid, then player i receives pay-off $V_i^j = x_i^j - B_i^j + \frac{1}{n} B_N^j$. By the rules of the multibidding game $B_i^1 = -B_i^2$ for every i, so bids may be negative, but the aggregated winning bid B_N^j is always non-negative.

The multibidding game achieves *budget balance* by construction, because the raised revenue by bids is entirely given back to participants. The social planner or some central authority does not need any positive or negative amount of money to operate it, therefore it is safe.

The other properties of the mechanism are studied assuming that agents behave strategically and form their bids as to maximize their expected payoff based on the information available to them. Their being ex ante identical, the symmetric Bayes-Nash equilibria (SBNE) of the game are considered. Therefore, the bid for a given project j is represented by $B^j(x_i^1, x_i^2)$ as a function of the personal characteristics whose form does not depend on the identity of the player. The expected utility for player i is defined as the expected value of V_i^j. The bidding function that maximizes players' expected utility will be called *optimal*.

Since submitted bids must add up to zero, agents are forced to report on their relative preferences between the two projects. The optimal bidding behavior of agents taking part in the multibidding game satisfies an appealing and intuitive property: it depends only on the difference between their private valuations for the two projects. That is, at equilibrium agents do report truthfully on their relative valuation of the projects.

Lemma 1 *In the SBNE of the multibidding game, the optimal bidding function depends only on the difference between private valuations for the two projects.*

Taking into account the result from Lemma 1, one can reformulate the problem at hand. For that, some more pieces of notation are needed. Let the difference between player i's private valuations be d_i with the following definition: $d_i = x_i^1 - x_i^2$. This new variable is random in general, since it is defined by the difference between two other random variables. Abusing notation slightly, denote its density by $f(d)$ and its cumulative distribution function by $F(d)$. Due to presentational considerations, first I study problems in which $f(d)$ is symmetric to the origin.[4] There does not appear any subindex on these objects, because they are common to every agent and correspond to a central variable.

With the bidding function for project j being $B^j(d_i)$ for every player i, the payoff that player i receives if project j obtains the largest aggregated bid can be rewritten as:

$$V_i^j\left(x_i^j, d_1, \ldots, d_n, B^j\right) = x_i^j - B^j(d_i) + \frac{1}{n}\sum_{i\in N} B^j(d_i).$$

Player 1's expected utility, when she happens to value project 1 by x_1^1, d_1 utility units more than project 2, and bids as if this difference were a value y_1, can be written in the following form:

$$v_1\left(x_1^1, x_1^2, d_1, y_1, B^1, B^2\right) =$$
$$= \int \cdots \int_{\substack{(d_2,\ldots,d_n) \text{ such that} \\ \text{project 1 wins}}} V_1^1\left(x_1^1, y_1, \ldots, d_n, B^1\right) \cdot f(d_2) \cdot \ldots \cdot f(d_n)\, dd_2 \ldots dd_n +$$
$$+ \int \cdots \int_{\substack{(d_2,\ldots,d_n) \text{ such that} \\ \text{project 2 wins}}} V_1^2\left(x_1^2, y_1, \ldots, d_n, B^2\right) \cdot f(d_2) \cdot \ldots \cdot f(d_n)\, dd_2 \ldots dd_n.$$

For simplicity, I shall write player i's expected utility as $v_i\left[x_i^1, d_i, B(y_i)\right]$, because x_i^1 and d_i give the individual valuations for both projects and by Lemma 1, given the bidding function, it is d_i that determines bids. Also, Lemma 1 combined with the complementarity of bids makes that a single function B can characterize the bidding behavior. This notation will be very helpful in the following analysis and for this reason let me reiterate the meaning of the above symbols. Player 1, exactly as the other $(n-1)$ players in the game, considers two possible results of the social decision procedure: either project 1 or project 2 will be carried out. The first one delivers x_1^1 units of utility to player 1 who must

[4] This assumption on the symmetry of the distribution is not crucial for any of my results, but makes explanations simpler. I comment on the consequencies of asymmetry in a separate section.

pay her bid, $B^1(y_1)$, for project 1 and will receive the nth part of the aggregated bid, $\frac{1}{n}\left[B^1(y_1)+\sum_{j\in N\setminus\{1\}}B^1(d_j)\right]$. Note that $B^1(y_1)$ can perfectly be a negative number, nevertheless I shall use the term *pay* when referring to monetary transactions according to bids. The expression for the expected utility involves $(n-1)$ integrals, because every agent is faced with the uncertainty captured by $(n-1)$ random variables, the differences between others' private valuations. The second term is to be interpreted in a similar way.

The characterization of the optimal bidding function can be enriched by some general results on its smoothness and increasing nature. The proof behind these intuitive facts uses standard arguments to be found, for example, in Fudenberg and Tirole (1991), but adapted to the multibidding game.

Lemma 2 *In the SBNE of the multibidding game, the optimal bidding function is continuous and strictly increasing.*

Thanks to the assumption on the symmetry of the underlying distribution, the optimal bidding function is also symmetric as it is shown in Lemma 3.

Lemma 3 *In the SBNE of the multibidding game, the optimal bidding function satisfies the following symmetry property:* $B^j(-d_i)=-B^j(d_i)$ *for every* j *and* d_i*.*

This result has a key role in deriving ex post efficient outcomes. It simplifies proofs and helps to compare the multibidding game to other mechanisms in the literature. Its impact is studied carefully in the next sections.

Taking into account the situations in the above examples, it seems natural to suppose that agents might abstain from participating in the bidding (decision making), but can not escape from the externalities, if such external effects exist. For example, villages and towns affected by the public project may wish not to exert influence on the choice of the project, but are still affected by both the positive and the negative consequences of the others' decision. The multibidding mechanism, however, has another appealing property which assures that agents cannot do better by staying out of the decision making process.

Proposition 1 *The multibidding mechanism is individually rational.*

The intuition behind the above result is that non-participation, as for bids and the collective choice of the project to be carried out, is equivalent to bidding zero. This bid, of course, will not, in general, be optimal. Moreover, the abstaining agent looses her part from the aggregated bid that is always non-negative in this mechanism.

Now, I start analyzing the efficiency properties of the multibidding mechanism with private information. For two players, one can compute the explicit form of the optimal bidding function in the multibidding mechanism; in the present set-up, it is always efficient. If there are more than two players in the game, efficiency is not guaranteed in general. However, the problem of inefficient decisions diminishes with a large number of players or a large degree of uncertainty.

3 The two-player case

Consider the situation in which a casino must be located in one of two cities; suppose these cities have no precise information concerning how the other values the project. When cities are asked individually for their preferences, they have incentives to exaggerate, not to report it truthfully. The multibidding mechanism can help to overcome this problem in the decision making process. In this example, the following interpretation is given to the previously defined variables:

- Project i: city i builds the casino.
- The differences between private valuations d_i show how city i's utility changes when city 1 gets the right to build the casino. Let $B(d_i)$ denote the optimal bidding function determining city i's bid for project 1.

When both x_1^1 and x_2^2 are positive, and $x_1^2 = x_2^1 = 0$, we have the case in which a desired object has to be allocated between two agents who experience no regret or envy when loosing. I shall refer to this case as the *classical case*.[5]

Now city 1, which experiences x_1^1 and d_1, and bids according to some function B at point y_1, has to maximize the following expression:

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{B(y_1)+B(d_2)\geq 0} \left\{x_1^1 - B(y_1) + \frac{1}{2}\left[B(y_1) + B(d_2)\right]\right\} \cdot f(d_2)\, dd_2 + \\ + \int_{B(y_1)+B(d_2)\leq 0} \left\{x_1^2 + B(y_1) - \frac{1}{2}\left[B(y_1) + B(d_2)\right]\right\} \cdot f(d_2)\, dd_2.$$

In this situation, the multibidding mechanism generates the ex post efficient outcomes; i.e., it chooses efficient projects that are socially optimal. In the classical case, it means that it assigns the object to the player who values it most.

Proposition 2 *In its SBNE with two players, the multibidding mechanism is efficient.*

[5]Note that the classical case enters in my setup if x_1^1 and x_2^2 have the same symmetric distribution, while x_1^2 and x_2^1 are degenerate random variables.

Symmetry of the optimal bidding function, its monotonicity, and the winning project's being chosen by the largest aggregated bid deliver this result. Intuitively, it is because the complemetary bids of the multibidding mechanism that extract information from participants on their relative private valuations between the projects. Since one of the two projects must be carried out by assumption, the absolute social impact of the projects is irrelevant for efficiency. Social welfare is maximized taking into account the sum of individual relative impacts that are revealed truthfully in the equilibrium aggregated bids.

The multibidding game is secure for participants too, because they can guarantee for themselves a minimum payoff by bidding the half of the difference between their private valuations for the two projects. Since the aggregate bid for the winning project is always non-negative, the utility level that players enjoy ex post is never less than the personal average of private valuations. The bidding function represented by a line with slope $\frac{1}{2}$ corresponds to these maximin strategies.

Efficiency, budget balance, and individual rationality are appealing properties, but one also might be interested in the explicit form of the optimal bidding function. This could be used in empirical work when one recovers private valuations from data on observed bids. Denote by d_M the median difference, defined by the difference that solves the following equality $F(d_M) = \frac{1}{2}$.[6]

Proposition 3 *In the SBNE of the multibidding game with two players, the optimal bidding function can be written as*

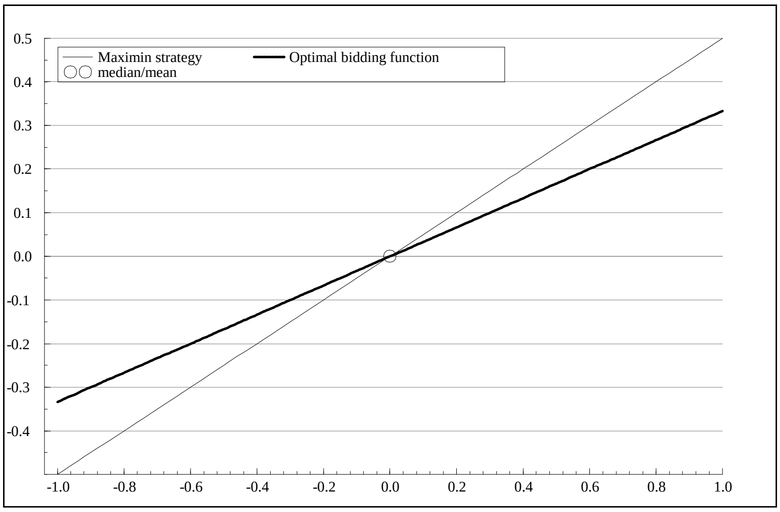
$$B(d_i) = \left\{ \begin{array}{cc} \frac{1}{2}d_i + \frac{1}{2}\left[1 - 2F(d_i)\right]^{-2} \cdot \int_{d_i}^{d_M} \left[1 - 2F(t)\right]^2 dt & \textit{if } d_i < d_M \\ \frac{d_i}{2} & \textit{if } d_i = d_M \\ \frac{1}{2}d_i - \frac{1}{2}\left[1 - 2F(d_i)\right]^{-2} \cdot \int_{d_M}^{d_i} \left[1 - 2F(t)\right]^2 dt & \textit{if } d_i > d_M \end{array} \right\}. \tag{1}$$

When considering SBNE, Proposition 3 shows that the above described maximin bidding behavior is only optimal at the median difference d_i. For d_i's above the median it is optimal to bid less *aggressively*, because bidding truthfully according to the optimal bidding function balances the probability of the preferred project to win and the utility loss due to paying bids. This maximizes agent i's expected utility, because with d_i increasing above the median level, the population that agent i should outbid in order to achieve a favorable outcome for herself is getting smaller. The intuition for values below the median is very similar and it also follows from the symmetry property of the optimal bidding function.

[6] Since the distribution of d_i is symmetric here, the median coincides with the expected value. But this is not the case in general as I discuss in Section 5.

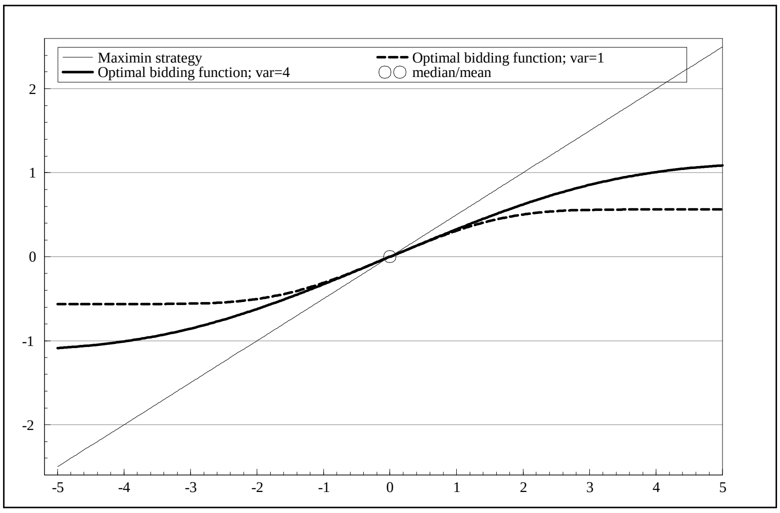
Before deriving result for the general n-player case, I consider some numerical examples which involve computing and plotting the optimal bidding function for two concrete distributions, the uniform and the normal. The uniform and the normal distributions, apart from their practical importance, play a crucial role in the general case.

Example 1 *The uniform distribution: agents attach the same likelihood to each value in the interval from which the differences between private valuations are drawn. When differences are distributed uniformly, $d_i \sim U[a;b]$, the mathematical form of the optimal bidding function can be simplified to $B(d_i) = \frac{1}{3}d_i + \frac{a+b}{12}$ for $d_i \in [a;b]$. Note that the function is linear. This feature is a property of the uniform distribution, because when player i increases her bid from $B(d_i)$ with one unit she outbids the same number of players independently on the original bid, $B(d_i)$. When the uniform distribution is symmetric abound 0, the optimal bidding function is proportional, and does not depend on the limits of the interval of possible differences. The slope is $\frac{1}{3}$ of the experienced difference. Graph 1 plots the optimal bidding function in the $U[-1;1]$ case. For reference the picture contains the $\frac{1}{2}d_i$ maximin line.*



(Graph 1. Optimal bidding function with uniform distribution and maximin strategies.)

Example 2 *The normal distribution. In this example I consider the standard normal distribution and another having mean zero and variance four. The optimal bidding functions cannot be put in a simple explicit form as in the previous example. Therefore, I represent them graphically. Graph 2 also contains the $\frac{1}{2}d_i$ maximin line for reference. As one can observe in both cases, the optimal bidding function equals zero when the difference between private valuations is zero, its slope increases and it gets closer to linear as the* *variance (uncertainty) increases.*



(Graph 2. Optimal bidding function with normal distributions and maximin strategies.)

4 Large groups

The construction of a casino may affect the welfare of a whole community formed by many agents. Therefore, it is important to explore the properties of the multibidding game in the presence of groups with cardinality larger than two. It turns out that whenever there are more then two participants in the bidding the characteristics of the SBNE of the mechanism related to efficiency change.

Lemma 4 *In its SBNE with $n > 2$, the multibidding mechanism is ex post efficient if and only if the optimal bidding function is proportional, i.e. $B(d_i) = \beta \cdot d_i$ with some parameter $\beta > 0$ for all $i \in N$.*

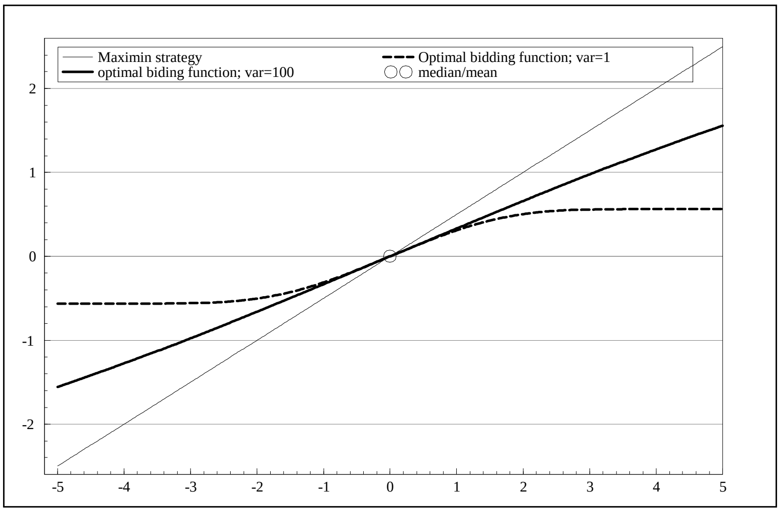
Efficiency of the multibidding mechanism cannot be guaranteed, in general, for any number of players. In the case of large groups, the efficiency requirement puts an important restriction on the admissible bidding function in equilibrium: it must be proportional to d_i.

Even if proportional functions are intuitive and easy to analyze, it turns out that they are suboptimal, in general. The reason behind this finding can be described as follows. Participants in the n-player case are face an aggregate of bids that can be considered as the bid of an imaginary player with a difference between her private valuations defined by $D = \sum_{j \in N \setminus \{i\}} d_i$. Knowing $f(d_i)$ the distribution of this aggregate can be characterized, being the sum of $(n-1)$ independent and identically-distributed random variables whose density I shall denote by $f_D(D)$. With a proportional bidding function this imaginary player bids $\beta \cdot D$ for project 1. For example, if each d_i is drawn from the normal distribution, then D will be distributed normally, too. And we have seen in the previous section that in that case the optimal bidding function is not proportional, not even linear.

Nevertheless, when n gets large, the distribution of D can be characterized by a very flat density function, since the variances of d_i add up. This distribution can also be considered as very close to a uniform. When this distribution can be approximated by a uniform distribution that is symmetric around zero, the multibidding mechanism can approximate ex post efficiency. Therefore, a proportional bidding function is not a *bad* choice whenever the number of participants is large enough. Proposition 4 and its proof make the above argument more rigorous.

Proposition 4 *In the SBNE of the multibidding game, when n is large, the optimal bidding function is close to a proportional function with slope $\frac{n}{4n-2}$.*

Graphs 3 delivers the graphical argument behind Proposition 4. It plots the optimal bidding function for the case with two players when the distribution of differences is normal with a large variance (100).[7] For reference it also contains the $\frac{1}{2}d_i$ line and the optimal bidding function computed with a standard normal distribution. One can observe that with the increase of the variance the bidding function in equilibrium gets close to linear, in particular to a proportional function with slope $\frac{1}{3}$.



(Graph 3. Optimal bidding function with normal distribution and maximin strategies.)

The intuition behind the result can be described in the following way: as the number of participants gets larger each agent faces higher uncertainty, because the sum of everybody else's bid, D, can obtain values from a larger set. In statistical terms, the variance of D

[7] The normal distribution is considered here, because by the central limit theorem the distribution of D gets close to normal with growing variance as n increases.

is getting larger. Instead of computing the exact distribution of D, agents might find satisfactory to approximate it by a uniform distribution. In the proof of Proposition 4 I show that the error of this approximation can be as small as one may require if the number of agents can grow arbitrary large. Proposition 5 gives the rate of convergence by showing the order of the approximation error under the condition that f has uniformly bounded third moments.

Proposition 5 *The error in the approximation (around zero) of the density of a sum of centered, independent and identically-distributed random variables that has uniformly bounded third moments, with a constant is of order* $n^{-\frac{1}{2}}$*.*

In the case of a uniform distribution that is symmetric around zero the optimal bidding function is proportional. Once Proposition 4 and Lemma 4 are combined, it is shown that the multibidding mechanism recovers efficiency if the number of affected parties (i.e. participants) is large. On the efficiency properties of the mechanism, I state the following two propositions.

Proposition 6 *In its SBNE, when* n *is large, the multibidding mechanism is close to efficient.*

Proposition 7 offers a result similar to the ones in Proposition 4 and Proposition 6 without the condition on n, the number of participants, being large, but with individual uncertainty of a very high degree. Technically speaking this means that the variance of the d_i is large. Therefore, the variance of the aggregate D is also very large. With this the multibidding mechanism can approximate efficiency also in cases with a small number of players that face big uncertainty.

Proposition 7 *In its SBNE, when uncertainty is large, the multibidding mechanism is close to efficient.*

A few comments on two practical features of the n-player model are now in order. The efficiency of the mechanism is obtained only in the limit, but in empirical situations one hardly finds an infinite number of participants. The following three points give support for the possible existence of efficient outcomes and suggest a method that agents might use in order to compute their *almost* optimal bidding function.

- Consider a finite number of participants. As shown in the proof of Proposition 4, if the optimal bidding function is linear, i.e. $B(d_i) = \beta \cdot d_i$, the slope coefficient, β, should solve the following equality for all d_i

$$\frac{-d_i \cdot f_D(-d_i)}{\left(\frac{1}{n}-1\right)+2\left(1-\frac{1}{n}\right)\cdot F_D(-d_i)-2d_i\cdot f_D(-d_i)}=\beta, \tag{2}$$

where the symbol $F_D(\cdot)$ stands for the accumulative distribution function of D. This is clearly impossible, in general. That is why the multibidding mechanism only reaches efficiency in the limit. Nevertheless, for large n, agents might bid proportionally, since the error they make decreases with n. On the other hand, the proportional bidding function is easy to apply and analyze.

For simplicity, denote the left-hand side of equation 2 by $b(d_i)$. Let us denote the largest and the smallest possible value of d_i by $d_{\max}$ and $d_{\min}$ respectively.[8] Now agent i can find the value for β that minimizes the mean squared error (MSE), defined as $MSE = \int_{d_{\min}}^{d_{\max}} [b(d_i) - \beta]^2 \cdot f(d_i)\, dd_i$. The minimization problem $\min_\beta MSE$ such that $\beta > 0$ implies that $\beta = \int_{d_{\min}}^{d_{\max}} b(d_i) \cdot f(d_i)\, dd_i = E[b(d_i)]$, where $E[\cdot]$ is the expected value operator.

- The proof of Proposition 4 shows that the error made by approximating the optimal bidding function by a proportional one diminishes as the number of participants grows. For efficiency, a large number of participants is needed. However, it is natural to ask how large is large. Even though I can not deliver an explicit formula for the optimal bidding function in the general n-player case, simulations have been performed and their results answer the above question.[9] Numerical simulations of the multibidding game also suggest that efficiency increases with the number of bidders (above two) in a continuos way. For the case in which uncertainty is captured by the uniform distribution, $U[-1;1]$, Table 1 shows the number of efficient decision as a function of the number of bidders.

n	2	3	5	10	20
efficient decisions	100%	98.6%	99.1%	99.5%	99.6%

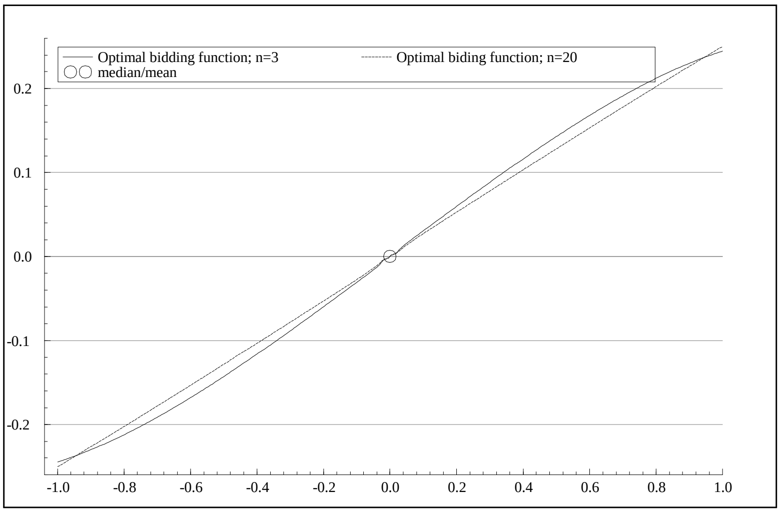
(Table 1. Number of efficient decisions as a function of group size in the $U[-1;1]$ case.)

One can observe that, even in the 3-player case, one with the highest number of inefficient decisions, approximately 98.6% of the decisions will maximize social welfare. Graph 4 plots the simulated optimal bidding function for the 3-player and 20-player cases in this example. It illustrates how the function looses curvature and

[8] The limits, $d_{\max}$ and $d_{\min}$, may very well be infinite.

[9] The simulation results have been generated using Ox version 2.20 (see Doornik, 1999), and are based on theoretical results that are presented in a subsection by the end of the appendix.

gets proportional with the increasing number of participants.



(Graph 4. Simulated optimal bidding functions for the $U[-1;1]$ case with 3 and 20 players.)

- One can also argue that the interpretation of the above assumptions can be changed in the following way: agents' prior beliefs might not coincide with the underlying true distributions. The results hold, as long as they are symmetric and identical for every participant in the model. This argument gives more field for the efficiency result in the n-player case: when agents expect in a symmetric manner that every state of the world is equally likely to occur, the distribution of D will be symmetric and uniform. In this case, agents will bid according to a proportional function in equilibrium. Therefore, ex post efficiency will be achieved.

5 Asymmetries and a dissolving partnership

The literature on mechanism design has discussed extensively the problem of dissolving partnerships. The problem is a classical one which has admitted efficient solutions under fairly general conditions. For a broad summary of the performance of simple mechanisms that one might use in such situations under uncertainty see McAfee (1992). The multi-bidding mechanism widens this list. The two-player problem serves as a reference point for further generalization. Moreover, this example will be useful in order to illustrate the importance of symmetry in prior beliefs. The assumption concerning the symmetry of the distribution of the difference between private valuations, d_i, is now relaxed and its consequences are studied.[10]

[10]Nevertheless, I keep the assumption of symmetry of the support of this distribution. The lack of this assumption would bring us to the case that is known as asymmetric auctions in the literature. At this point of the study of the multibidding game I wish to concentrate on other features of the mechanism and keep this topic for further research.

When a marriage or, in general, a partnership breaks down there are usually indivisible objects to be allocated among two agents. For technical reasons, the literature on mechanism design, and closer the literature on auction theory, typically considers a single object. Using now this nomenclature, two parties and two projects exist: under one party 1 receives the object, while under the other party 2 gets it. I shall assume that players have private valuations over these projects and the social planner wishes to allocate the object taking into account social welfare and is not interested in raising revenue.

Let me now consider two parties and an indivisible good that has to be allocated among them. In this section, we shall use the multibidding mechanism to solve the problem. For this reason, the following interpretation is given to the variables:

- Project i: player i receives the object.

As for the differences between private valuations, in the two-player case one can proceed in two ways to be called the symmetric case and the asymmetric one due to the different meaning of the bidding function in them. I introduce the following piece of notation: f^* is a density function such that $f(-d) = f^*(d)$ for all d. The respective cumulative distribution function is F^*. $B^*(d_i)$ denotes the optimal bidding function in the case of $f^*(d)$ being the density of the underlying distribution and $F^*(d)$ its distribution function. In other words, if $B(\cdot)$ represents bids for project 1, then $B^*(\cdot)$ denotes bids for its alternative computed in the problem where project names are reversed, and vice versa.

Lemma 5 *In the SBNE of the multibidding game, the optimal bidding function satisfies the following property:* $B^*(-d_i) = -B(d_i)$ *for every* d_i.

- The asymmetric case arises once one defines the differences between private valuations in the following way: $d_1 = x_1^1 - x_1^2$ and $d_2 = x_2^2 - x_2^1$. Hence, d_i shows how agent i's utility changes when she gets the object. Therefore, the optimal bidding function $B(d_i)$ can be interpreted as player i's bid for having the object. I shall assume that the distributions of these two differences coincide and can be characterized by functions $f(d)$ and $F(d)$. However with this, players value the projects in an asymmetric, in fact opposite, way. The bidding function (1) presented in Proposition 3 is the optimal bidding function in the asymmetric case for any underlying distribution characterizing uncertainty. This guarantees ex post efficiency in general.

- The symmetric case follows from the model specification according to which $d_1 = x_1^1 - x_1^2$ and $d_2 = x_2^1 - x_2^2$. With this, the optimal bidding function $B(d_i)$ can be interpreted as player i's bid for the first project in equilibrium. Similarly, in the asymmetric case, consider situations in which the distributions of d_1 and d_2 coincide, and can be characterized by the density function $f(d)$ and the cumulative distribution function $F(d)$. The name *symmetric* is due to the latter assumption, since now players value the projects in the same manner, according to the same underlying distribution that does not need to be symmetric. The symmetry of prior belief on d_i is crucial for ex post efficiency in this case. If prior beliefs follow an asymmetric distribution, then inefficient decisions may occur in the symmetric case. Proposition 8 and 9 analyze this problem.

In the symmetric case, players tend to prefer the same project and seem not to be as antagonistically opposed as in the asymmetric case. This situation may arise, for example, when the two affected parties share the same opinion on the allocation of the indivisible object in question. That is, they tend to value the projects in the same way, according to the same underlying distribution. Based on Lemma 3 it is easy to derive the explicit form of the optimal bidding function, and I can state the symmetric version of Proposition 3.

Proposition 8 *In the SBNE of the multibidding game with two players, the optimal bidding function can be written as*

$$B(d_1) = \left\{ \begin{array}{cl} \frac{1}{2}d_1 + \frac{1}{2}\left[1-2F^*(d_1)\right]^{-2} \cdot \int_{d_1}^{d_M^*} \left[1-2F^*(t)\right]^2 dt & \text{if } d_1 < d_M^* \\ \frac{d_1}{2} & \text{if } d_1 = d_M^* \\ \frac{1}{2}d_1 - \frac{1}{2}\left[1-2F^*(d_1)\right]^{-2} \cdot \int_{d_M^*}^{d_1} \left[1-2F^*(t)\right]^2 dt & \text{if } d_1 > d_M^* \end{array} \right\}.$$

Remember that by definition $F^*(d_M^*) = \frac{1}{2}$. Note that the distinction between the symmetric and the asymmetric cases becomes superfluous whenever the underlying distribution of differences in valuations is symmetric. This intuitive fact makes that the bidding functions presented in Graphs 1-3 are optimal both in the symmetric and asymmetric set-up.

The result on the optimal bidding function in the multibidding game shares some interesting features with the cake-cutting mechanism (CCM) studied in McAfee (1992).[11] In the CCM, players bid their true valuations at the median. In the multibidding game, at the median players bid half of the difference between their valuations. This not being the whole truth can be intuitively explained by the rules of the multibidding mechanism, because players are forced to bid over two projects and bids must sum up to zero. Below the median value, players overbid in the sense that $B(d_i)$ is larger than the half of the difference between the private valuations. While above the median they underbid.

[11] In the cake-cutting mechanism, one party proposes a division and the other party chooses one of the parts of the division. This mechanism can be adapted to the indivisible case when money is available in the economy. For more details check McAfee (1992).

Nevertheless, there is an important difference between the CCM and the multibidding mechanism: the latter treats players symmetrically and precisely because ex post efficiency can be achieved. The CCM, distinguishing the roles of proposer and chooser, turns out to be "ex post inefficient, and in an unusual way" [McAfee (1992)].

In the (symmetric) case, when players bid for the same project according to the same bidding function, this may cause the loss of ex-post social efficiency. As shown previously, this problem is absent when players bid for opposite projects using the same bidding function. The next proposition states that for ex-post efficiency, in the symmetric case, a certain condition on the symmetry of the optimal bidding function must hold. This condition requires the symmetry of the distribution of the prior beliefs.

Proposition 9 *In its SBNE with two players, the multibidding mechanism is efficient if and only if the prior distribution is symmetric, that is if and only if the following condition holds:* $B(-d_i) = -B(d_i)$ *for every* d_i *and every* i.

Section 4 showed that in situations with more than two players the multibidding game can only deliver ex post efficient decisions if players bid according to a proportional function in equilibrium. Once the original assumption of symmetry of the underlying density function is relaxed, an extra condition is needed to ensure proportionality in the n-player case. The increasing number of bidders increases uncertainty and makes the optimal bidding function flatter, closer to linear in the model. With this, the number of ex post efficient decisions also increases. However a constant term in the bidding function works against this improvement and makes inefficient decisions persist even with very large number of players. As shown in the proofs of the propositions for the n-player case, the expected value of the aggregate D must be zero for results to hold. This condition is satisfied when the distribution of d is symmetric; i.e., when agents value the two project equal in expected terms, since this implies that the expected value of d and also D is zero.

6 Conclusions

I have examined the problem of choosing a project efficiently by a group of agents, and studied the theoretical performance of the multibidding mechanism in situations in which agents may hold private information. My analysis is embedded in a general model with any number, n, of players and any number, m, of projects. Therefore, in the present work, I determined the properties of equilibria in the case of two available projects and risk-neutral players. The complexity arising when more than two projects or risk-aversion appears because of the rules of the multibidding game, expected utilities depend on more than one variable. When two projects exist, agents' expected utility depends on the two private valuations, too, but the dimension of the problem can be reduced by one. As has been shown, it is enough to know the difference between those private valuations in order to be able to determine the optimal bidding behavior. The multibidding mechanism is always efficient in the two-player two-project case with the above restriction, and with the symmetry of prior distributions or asymmetry of players, while efficiency is tied to more conditions when there are more players: the number of agents must be large or (with a similar intuition behind) uncertainty must be large with zero expected value, in order to achieve efficient outcomes. Because of presentational considerations, a continuum of possible valuations has been used, but the results, with the proper modification, hold in the discrete case too.

It is important to bear in mind that in the analysis attention was focused on symmetric Bayes-Nash equilibria; i.e. agents face the same uncertainty and act according to the same optimal bidding function. The appealing features of the multibidding mechanism without uncertainty, and under uncertainty with two projects and risk neutral agents, make it a powerful tool for choosing an efficient project by some set of players in the presence of a public good and/or externalities. The mechanism is simple and can be easily understood by agents even in the most general $n \times m$ case. Determining the properties of its equilibria in the general case is a topic for further research.

Beside its theoretical performance, both with and without uncertainty, the multibidding game has also appealing empirical properties. Pérez-Castrillo and Veszteg (2004) report results from the experimental laboratory on the mechanism presented here. In terms of efficiency, the multibidding game selects the ex post efficient project in roughly three quarters of the cases across four experimental treatments. In line with the theoretical predictions, the number of efficient decisions was larger when individuals were paired than when they formed groups of larger size. Also, the largest part of the subject pool formed their bids according to the theoretical Bayes-Nash bidding behavior.

7 Appendix

The appendix contains the formal proof of all the results in the paper in the order as they appear in the text.

Proof of Lemma 1. Consider the following notation: agent 1 experiences (x_1^1, x_1^2) and bids for project 1 according to some function $B^1 = -B^2$ at (y_1^1, y_1^2). The other agents have private valuations $\left(x_{-1}^1, x_{-1}^2\right) = \left[\left(x_2^1, x_3^1, \ldots\right), \left(x_2^2, x_3^2, \ldots\right)\right]$ and bid truthfully using the same function B^1. The distribution of the vector x_{-1}^j can be characterized by the density f_j which is the joint density of the others' valuations for project j. The expected utility for agent 1 can be written as:

$$v_1\left[x_1^1, x_1^2, B^1\left(y_1^1, y_1^2\right)\right] =$$

$$= \int\int_{\substack{\left(x_{-1}^1, x_{-1}^2\right) \text{ such that} \\ \text{project1 wins}}} \left[x_1^1 - B^1\left(y_1^1, y_1^2\right) + \frac{1}{n}B^1\left(y_1^1, y_1^2\right) + \frac{1}{n}\sum_{i\in N\setminus\{1\}} B^1\left(x_i^1, x_i^2\right)\right] \cdot$$
$$\cdot f_1\left(x_{-1}^1\right)\cdot f_2\left(x_{-1}^2\right) dx_{-1}^1 dx_{-1}^2 +$$

$$+ \int\int_{\substack{\left(x_{-1}^1, x_{-1}^2\right) \text{ such that} \\ \text{project2 wins}}} \left[x_1^2 + B^1\left(y_1^1, y_1^2\right) - \frac{1}{n}B^1\left(y_1^1, y_1^2\right) - \frac{1}{n}\sum_{i\in N\setminus\{1\}} B^1\left(x_i^1, x_i^2\right)\right] \cdot$$
$$\cdot f_1\left(x_{-1}^1\right)\cdot f_2\left(x_{-1}^2\right) dx_{-1}^1 dx_{-1}^2$$

Now consider the case in which agent 1's private values are $(x_1^1+\delta, x_1^2+\delta)$ where δ has a constant real value. In order to prove Lemma 1 it is enough to show that

$$\frac{\partial v_1\left[x_1^1, x_1^2, B^1\left(y_1^1, y_1^2\right)\right]}{\partial y_1^j} = \frac{\partial v_1\left[x_1^1+\delta, x_1^2+\delta, B^1\left(y_1^1, y_1^2\right)\right]}{\partial y_1^j} \tag{3}$$

for $j = 1, 2$. Note that

$$v_1\left[x_1^1+\delta, x_1^2+\delta, B^1\left(y_1^1, y_1^2\right)\right] =$$

$$= \int\int_{\substack{\left(x_{-1}^1, x_{-1}^2\right) \text{ such that} \\ \text{project1 wins}}} \left[x_1^1 + \delta - B^1\left(y_1^1, y_1^2\right) + \frac{1}{n}B^1\left(y_1^1, y_1^2\right) + \frac{1}{n}\sum_{i\in N\setminus\{1\}} B^1\left(x_i^1, x_i^2\right)\right] \cdot$$
$$\cdot f_1\left(x_{-1}^1\right)\cdot f_2\left(x_{-1}^2\right) dx_{-1}^1 dx_{-1}^2 +$$
$$+ \int\int_{\substack{\left(x_{-1}^1, x_{-1}^2\right) \text{ such that} \\ \text{project2 wins}}} \left[x_1^2 + \delta + B^1\left(y_1^1, y_1^2\right) - \frac{1}{n}B^1\left(y_1^1, y_1^2\right) - \frac{1}{n}\sum_{i\in N\setminus\{1\}} B^1\left(x_i^1, x_i^2\right)\right] \cdot$$
$$\cdot f_1\left(x_{-1}^1\right)\cdot f_2\left(x_{-1}^2\right) dx_{-1}^1 dx_{-1}^2 =$$
$$= v_1\left[x_1^1, x_1^2, B^1\left(y_1^1, y_1^2\right)\right] + \delta.$$

Taking into account the first and the last expression in the equality above (3) follows immediately. ■

Proof of Lemma 2. Let us prove first that the optimal bidding function is increasing.

Note that for project 1 to be the winning project I must have a non-negative aggregated bid for project 1; i.e., $B(y_1) + \sum_{i\in N\setminus\{1\}} B(d_i) \geq 0$. Player 1's expected utility can be written in general as

$$v_1\left[x_1^1,d_1,B\left(y_1\right)\right]=\int\cdots\int_{\substack{(d_2,\ldots,d_n)\text{ such that}\\ \text{project1 wins}}}\left[x_1^1-B\left(y_1\right)+\frac{1}{n}B\left(y_1\right)+\frac{1}{n}\sum_{i\in N\backslash\{1\}}B\left(d_i\right)\right]\cdot$$
$$\cdot f\left(d_2\right)\cdot\ldots\cdot f\left(d_n\right)dd_2\ldots dd_n+$$
$$+\int\cdots\int_{\substack{(d_2,\ldots,d_n)\text{ such that}\\ \text{project2 wins}}}\left[x_1^2+B\left(y_1\right)-\frac{1}{n}B\left(y_1\right)-\frac{1}{n}\sum_{i\in N\backslash\{1\}}B\left(d_i\right)\right]\cdot$$
$$\cdot f\left(d_2\right)\cdot\ldots\cdot f\left(d_n\right)dd_2\ldots dd_n.$$

Since B is the optimal bidding function, for any d_1 and d_1^* such that $d_1>d_1^*$ I have that

$$\begin{aligned}v_1\left[x_1^1,d_1,B\left(d_1\right)\right] &\geq v_1\left[x_1^1,d_1,B\left(d_1^*\right)\right];\\ v_1\left[x_1^1,d_1^*,B\left(d_1^*\right)\right] &\geq v_1\left[x_1^1,d_1^*,B\left(d_1\right)\right].\end{aligned}$$

And therefore

$$v_1\left[x_1^1,d_1,B\left(d_1\right)\right]-v_1\left[x_1^1,d_1^*,B\left(d_1\right)\right]\geq v_1\left[x_1^1,d_1,B\left(d_1^*\right)\right]-v_1\left[x_1^1,d_1^*,B\left(d_1^*\right)\right]. \tag{4}$$

For the sake of this proof let us normalize player 1's private valuation such that $d_1=x_1^1$, $(0=x_1^2)$ and $d_2=x_2^2$, $(0=x_2^1)$. This will not effect the generality of my results since this normalization can be done by adding/subtracting the same constant from both sides in inequality 4. Let us substitute the expected utilities with their form in integrals and simplify the result.

$$\int\cdots\int_{B(d_1)+\sum_{i\in N\backslash\{1\}}B(d_i)\geq 0}\left(d_1-d_1^*\right)\cdot f\left(d_2\right)\cdot\ldots\cdot f\left(d_n\right)dd_2\ldots dd_n\geq$$
$$\geq\int\cdots\int_{B\left(d_1^*\right)+\sum_{i\in N\backslash\{1\}}B(d_i)\geq 0}\left(d_1-d_1^*\right)\cdot f\left(d_2\right)\cdot\ldots\cdot f\left(d_n\right)dd_2\ldots dd_n$$

For this inequality to hold I must have $B\left(d_1\right)\geq B\left(d_1^*\right)$ and this completes the first part of the proof.

Strict monotonicity and continuity can be proven using a standard indirect argument following Fudenberg and Tirole (1991). I only explain the idea of the proof here.

Strict monotonicity: suppose that there is an atom at b in the bidding function, that is $pr\left[B\left(d_j\right)=b\right]>0$ for some agent j. In this case agent i would assign probability 0 to the interval $[b-\varepsilon;b)$ for some $\varepsilon>0$, and she bids just above b. But then agent j with a difference d_j such that $B\left(d_j\right)=b$, would be better off bidding $b-\varepsilon$, as this does not reduce the probability of winning, but does reduce cost. Therefore there cannot be an atom at b.

Continuity: if B is discontinuous I can find b' and $b''(> b')$ such that $pr\{B(d_j) \in [b'; b'']\} = 0$, while there exist d_j^* and $\varepsilon \geq 0$ for which $B(d_j^*) = b'' + \varepsilon$. In this case, agent i strictly prefers bidding b' to any other bid in $(b'; b'')$, since doing so does not reduce the probability of winning, but does reduce cost. But then agent j's choice of quitting at b'', or just beyond, is not optimal when she experiences d_j^*. Therefore B is continuous. ■

Proof of Lemma 3. I shall omit the superindex from the optimal bidding function in the proof, since $B^1(d_i) = -B^2(d_i)$ holds for every d_i. Suppose that agent i experiences private valuations with a difference of $d_i = x_i^1 - x_i^2$. Her bid for project 1 in the equilibrium can be computed according to the optimal bidding function and will be equal to $B(d_i)$. Due to the rules of the multibidding mechanism, in particular to the fact that bids must sum up to zero, with this her bid for project 2 is $-B(d_i)$. Now I can consider situations in which for player 1 it is more convenient to compute her bid for project 2 first, i.e. to take into account $d_i^* = x_i^2 - x_i^1 = -d_i$. Of course, equilibrium bids can not change with the above technicality, therefore $B^*(-d_i) = B^*(d_i^*) = -B(d_i)$. Since by symmetry the density functions of d_i and d_i^* coincide, we have for bidding functions that $B = B^*$. That is $B(-d_i) = -B(d_i)$. ■

Proof of Proposition 1. Consider agent i's expected payoff when her type is d_i. If she bids according to the optimal bidding function this quantity is equal to $v_i[x_i^1, d_i, B(d_i)]$. When agent i does not wish to influence the choice of the winning project she can bid 0, and with it obtain $v_i(x_i^1, d_i, 0)$ in expected terms. For any d_i by definition I have that $v_i[x_i^1, d_i, B(d_i)] \geq v_i(x_i^1, d_i, 0)$. With zero bid agent i does not affect the choice of the winning project, but does receive her part from the aggregated winning bid that is non-negative by the rules of the multibidding mechanism. If $v_i^a(x_i^1, d_i)$ is agent i's expected utility when she stays out of the process, then for any d_i I must have $v_i[x_i^1, d_i, B(d_i)] \geq v_i^a(x_i^1, d_i)$. That is the multibidding mechanism is individually rational. ■

Proof of Proposition 2. This is a direct consequence of the fact that the optimal bidding function is strictly increasing. To see this, consider the following table that describes a two-player situation in general with the notation introduced in the text before.

	project 1	project 2	d_i
player 1	x_1^1	x_1^2	$x_1^1 - x_1^2$
player 2	x_2^1	x_2^2	$x_2^1 - x_2^2$
$\sum$	$x_1^1 + x_2^1$	$x_1^2 + x_2^2$	$*$

Note that for ex post efficiency I need project 1 to win if and only if $x_1^1 + x_2^1 \geq x_1^2 + x_2^2$. That is $x_1^1 - x_1^2 + x_2^1 - x_2^2 \geq 0$, or $d_1 + d_2 \geq 0$. The above requirement is met since the optimal bidding function is strictly increasing and symmetric: $d_1 + d_2 \geq 0 \leftrightarrow d_1 \geq -d_2$ $\leftrightarrow B(d_1) \geq B(-d_2) \leftrightarrow B(d_1) \geq -B(d_2) \leftrightarrow B(d_1) + B(d_2) \geq 0 \leftrightarrow$ Project 1 wins. ■

Proof of Proposition 3. By result from Lemma 2 project 1 wins if $B(d_1) \geq B(-d_2)$, that is $d_1 \geq -d_2$. Therefore project 1 wins with probability $pr(d_1 \geq -d_2) = pr(-d_1 \leq d_2) = 1 - F(-d_1)$. Due to the assumption on the symmetry of the underlying density function the density of d_2 and $-d_2$ coincide. Now let us find the expected utility for player 1 that experiences $d_1 (= x_1^1 - x_1^2)$ and bids according to y_1 using the function B:

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{x_L}^{y_1} \left\{x_1^1 - B(y_1) + \frac{1}{2}\left[B(y_1) + B(d_2)\right]\right\} f(-d_2)\, d(-d_2) +$$
$$+ \int_{y_1}^{x_H} \left\{x_1^2 + B(y_1) - \frac{1}{2}\left[B(y_1) + B(d_2)\right]\right\} f(-d_2)\, d(-d_2).$$

By the symmetry property of the optimal bidding function one can write:

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{x_L}^{y_1} \left\{x_1^1 - \frac{1}{2}B(y_1) - \frac{1}{2}B(-d_2)\right\} f(-d_2)\, d(-d_2) +$$
$$+ \int_{y_1}^{x_H} \left\{x_1^2 + \frac{1}{2}B(y_1) + \frac{1}{2}B(-d_2)\right\} f(-d_2)\, d(-d_2).$$

In order to simplify the above expression let us use the following notation: $d_2^* = -d_2$.This will also help to interpret the proof in the case when I relax the assumption on the symmetry of f in Section 6.

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{x_L}^{y_1} \left[x_1^1 - \frac{1}{2}B(y_1) - \frac{1}{2}B(d_2^*)\right] f(d_2^*)\, dd_2^* + \qquad (5)$$
$$+ \int_{y_1}^{x_H} \left[x_1^2 + \frac{1}{2}B(y_1) + \frac{1}{2}B(d_2^*)\right] f(d_2^*)\, dd_2^*$$

Agents are supposed to maximize their expected utility in the bidding. The first order condition of the problem is derived below.

$$\frac{\partial}{\partial y_1} v_1\left[x_1^1, d_1, B(y_1)\right] = -\frac{1}{2}\int_{x_L}^{y_1} B'(y_1) f(d_2^*)\, dd_2^* + \left[x_1^1 - \frac{1}{2}B(y_1) - \frac{1}{2}B(y_1)\right] f(y_1) +$$
$$+\frac{1}{2}\int_{y_1}^{x_H} B'(y_1) f(d_2^*)\, dd_2^* - \left[x_1^2 + \frac{1}{2}B(y_1) + \frac{1}{2}B(y_1)\right] f(y_1) =$$
$$= -\frac{1}{2}B'(y_1) F(y_1) + \frac{1}{2}B'(y_1)\left[1 - F(y_1)\right] + \left[x_1^1 - x_1^2 - B(y_1) - B(y_1)\right] f(y_1) =$$
$$= \left[\frac{1}{2} - F(y_1)\right] B'(y_1) - 2B(y_1) f(y_1) + d_1 f(y_1) = 0$$

The optimal bidding function must solve the above differential equation for $y_1 = d_1$.

$$\left[\frac{1}{2} - F(d_1)\right] B'(d_1) - 2B(d_1) f(d_1) + d_1 f(d_1) = 0 \qquad (6)$$

If $F(d_1) = \frac{1}{2}$ then $B(d_1) = \frac{d_1}{2}$. If $F(d_1) \neq \frac{1}{2}$ we have that

$$B'(d_1) - 2\frac{f(d_1)}{\frac{1}{2} - F(d_1)} B(d_1) + d_1 \frac{f(d_1)}{\frac{1}{2} - F(d_1)} = 0.$$

Let us introduce the notation $A(d_1) = \frac{f(d_1)}{\frac{1}{2} - F(d_1)}$ and $x \in [x_L, x_H]$. The latter identifies the lowest and the largest admissible value for x. The differential equation and its general solution can be written now as

$$B'(d_1) - 2A(d_1) B(d_1) + d_1 A(d_1) = 0,$$
$$B(d_1) = \exp\left[2\int_x^{d_1} A(t)\, dt\right] \cdot \left(\eta - \int_x^{d_1} \left\{ tA(t) \cdot \exp\left[-2\int_x^t A(s)\, ds\right]\right\} dt\right).$$

Note that the integrals in the solution might include a difference such that $A(d_1)$ is not defined, therefore x must be carefully chosen. This parameter along with η can be fixed taking into account that the optimal bidding function must be continuous and strictly increasing.

One can check that the following function is the optimal bidding function in this problem:

$$B(d_1) = \left\{ \begin{array}{cl} \frac{1}{2}d_1 + \frac{1}{2}\left[1 - 2F(d_1)\right]^{-2} \cdot \int_{d_1}^{d_M} \left[1 - 2F(t)\right]^2 dt & \text{if } d_1 < d_M \\ \frac{d_1}{2} & \text{if } d_1 = d_M \\ \frac{1}{2}d_1 - \frac{1}{2}\left[1 - 2F(d_1)\right]^{-2} \cdot \int_{d_M}^{d_1} \left[1 - 2F(t)\right]^2 dt & \text{if } d_1 > d_M \end{array} \right\}. \tag{7}$$

To do so note that the following holds. For $d_1 > d_M$ fix some $x > d_M$ and choose η such that $B'(d_1) > 0$.

$$B(d_1) = \frac{1}{2}d_1 + \left[1 - 2F(d_1)\right]^{-2} \cdot \left\{\eta - \frac{1}{2}x_H + \frac{1}{2}\int_{d_1}^{x_H} \left[1 - 2F(t)\right]^2 dt\right\}$$

Take $\eta = \frac{1}{2}x_H - \frac{1}{2}\int_{d_M}^{x_H} \left[1 - 2F(t)\right]^2 dt$. It exists, it is finite, it does not depend on d_1 and guarantees the properties that I require from $B(d_1)$. In particular, the optimal bidding function needs to be continuous, therefore the above proposed value for η is unique. To see this note that according to (7) discontinuity may occur at the median, and also that η is a constant shifting parameter that allows us to move the optimal bidding function for all $d_1 < d_M$ in order to reach continuity at d_M. For $d_1 < d_M$ fix some $x < d_M$ and choose η such that $B'(d_1) > 0$.

$$B(d_1) = \frac{1}{2}d_1 + \left[1 - 2F(d_1)\right]^{-2} \cdot \left\{\eta - \frac{1}{2}x_L - \frac{1}{2}\int_{x_L}^{d_1} \left[1 - 2F(t)\right]^2 dt\right\}$$

Now take $\eta = \frac{1}{2}x_L + \frac{1}{2}\int_{x_L}^{d_M} \left[1 - 2F(t)\right]^2 dt$. It exists, it is finite, it does not depend on d_1 and guarantees the properties that I require from $B(d_1)$. As in below the median, the proposed value for η is unique here, too. These parameter values give the expression in equation 7 that completes the proof. Note that equation 6 and the monotonicity of the optimal bidding function give both upper and lower bounds for the bid $B(d_1)$ once d_1 is fixed:

$$B(d_1) \in \left\{ \begin{array}{cc} \left[\frac{d_1}{2}; \frac{d_M}{2}\right] & \text{if } d_1 < d_M \\ \left[\frac{d_M}{2}\right] & \text{if } d_1 = d_M \\ \left[\frac{d_M}{2}; \frac{d_1}{2}\right] & \text{if } d_1 > d_M \end{array} \right\}.$$

This result will be useful in the case with large groups. ■

Proof of Lemma 4. For ex post efficiency I need the aggregated optimal bid function to be a increasing strictly monotone function of the aggregated true valuations.

If the optimal bid function is proportional, $B(d_i) = \beta d_i$, this is the case, since $\sum_{i \in N} B(d_i) = \beta \sum_{i \in N} d_i$ holds. I already know that $\beta > 0$, since the optimal bidding function is strictly increasing.

In order to show the other implication consider the following. Suppose that I have $\sum_{i \in N} B(d_i) = B$ for some vector d with $\sum_{i \in N} d_i = A$ where A and B are some real numbers. Now let the valuation change for some players i_1 and i_2 such that $d^*_{i_1} = d_{i_1} + \Delta$, while $d^*_{i_2} = d_{i_2} - \Delta$. Therefore $\sum_{i \in N} d^*_i = A$. For the result to be ex post efficient I need the aggregated bid to remain unchanged. To see this consider the following inequalities implied by the ex post efficiency requirement:

$$\begin{aligned} \sum_{i \in N} d_i &\geq \sum_{i \in N} d^*_i = \sum_{i \in N} d_i \Leftrightarrow \sum_{i \in N} B(d_i) \geq \sum_{i \in N} B(d^*_i), \\ \sum_{i \in N} d_i &\leq \sum_{i \in N} d^*_i = \sum_{i \in N} d_i \Leftrightarrow \sum_{i \in N} B(d_i) \leq \sum_{i \in N} B(d^*_i), \end{aligned}$$

that is

$$\sum_{i \in N} d_i = \sum_{i \in N} d^*_i \Leftrightarrow \sum_{i \in N} B(d_i) = \sum_{i \in N} B(d^*_i).$$

Having the above results I can write

$$\begin{aligned} B\left(d^*_{i_1}\right) + B\left(d^*_{i_2}\right) + \sum_{i \in N \setminus \{i_1, i_2\}} B(d_i) - \sum_{i \in N} B(d_i) &= 0, \\ B(d_{i_1} + \Delta) + B(d_{i_2} - \Delta) + \sum_{i \in N \setminus \{i_1, i_2\}} B(d_i) - \sum_{i \in N} B(d_i) &= 0, \\ B(d_{i_1} + \Delta) - B(d_{i_1}) + B(d_{i_2} - \Delta) - B(d_{i_2}) &= 0, \\ \frac{B(d_{i_1} + \Delta) - B(d_{i_1})}{\Delta} &= \frac{B(d_{i_2}) - B(d_{i_2} - \Delta)}{\Delta}, \end{aligned}$$

for d_{i_1} and d_{i_2}, and all Δ. I can consider $\Delta \to 0$. The above requirement then says that $B'(d_{i_1}) = B'(d_{i_2})$ for d_{i_1} and d_{i_2}. Precisely this means that the optimal bidding function must be linear. Now let us argument that the constant term in this linear function must be equal to zero. If the mechanism is ex post efficient then $\sum_{i\in N} B(d_i) = n\alpha + \beta\sum_{i\in N} d_i \geq 0$ iff $\sum_{i\in N} d_i \geq 0$. ■

Proof of Proposition 4. Consider Player 1's expected utility with $B(d_i) = \beta d_i$. Since Project 1 is chosen if $\sum_{i\in N\setminus\{1\}} d_i = D \geq -y_1$,

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{D\geq -y_1}\left[x_1^1 + \left(\frac{1}{n}-1\right)\beta y_1 + \frac{1}{n}\beta D\right]\cdot f_D(D)\,dD +$$
$$+\int_{D<-y_1}\left[x_1^2 + \left(1-\frac{1}{n}\right)\beta y_1 - \frac{1}{n}\beta D\right]\cdot f_D(D)\,dD,$$

where $f_D(D)$ is the density function of the aggregate D. Note that $D\in[D_{\min}; D_{\max}]$ with some lower, $D_{\min}$, and upper bound, $D_{\max}$, therefore:

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{-y_1}^{D_{\max}}\left[x_1^1 + \left(\frac{1}{n}-1\right)\beta y_1 + \frac{1}{n}\beta D\right]\cdot f_D(D)\,dD +$$
$$+\int_{D_{\min}}^{-y_1}\left[x_1^2 + \left(1-\frac{1}{n}\right)\beta y_1 - \frac{1}{n}\beta D\right]\cdot f_D(D)\,dD.$$

$$\frac{\partial}{\partial y_1}v_1\left[x_1^1, d_1, B(y_1)\right] = \left(\frac{1}{n}-1\right)\beta\cdot\int_{-y_1}^{D_{\max}} f_D(D)\,dD + \left(1-\frac{1}{n}\right)\beta\cdot\int_{D_{\min}}^{-y_1} f_D(D)\,dD +$$
$$+\left[d_1 + 2\left(\frac{1}{n}-1\right)\beta y_1 - \frac{2}{n}\beta y_1\right]\cdot f_D(-y_1).$$

In the equilibrium the following equality is required to hold for every d_1:

$$\left(\frac{1}{n}-1\right)\beta\cdot\int_{-d_1}^{D_{\max}} f_D(D)\,dD + \left(1-\frac{1}{n}\right)\beta\cdot\int_{D_{\min}}^{-d_1} f_D(D)\,dD +$$
$$+\left[d_1 + 2\left(\frac{1}{n}-1\right)\beta d_1 - \frac{2}{n}\beta d_1\right]\cdot f_D(-d_1) = 0.$$

The expression can be put in a different way.

$$\beta\left[\left(\frac{1}{n}-1\right) + 2\left(1-\frac{1}{n}\right)\cdot F_D(-d_1) - 2d_1\cdot f_D(-d_1)\right] + d_1\cdot f_D(-d_1) = 0.$$

This expression, in general for any F_D and f_D, cannot be set to be equal to zero for all values of d_1 by choosing a constant value for β. I already know that the optimal bidding function is strictly increasing which can be translated into a strictly positive β in the proportional case. Nevertheless, if the functions F_D and f_D belonged to the uniform distribution over some interval $[-a; a]$ I would have $\beta = \frac{n}{4n-2}$. In other words, if the distribution of D is uniform with expected value zero, the optimal bidding function is proportional, hence ex post efficiency is achieved. This requirement is met in the special case of symmetric distributions. The interval, $[-a; a]$, is symmetric to zero by assumption, since D must have expected value zero. Since D is the sum of iid random variables as n gets very large it converges to a normally distributed variable whose expected value is zero and whose variance tends to infinity. Now let us argue that, when n is large, agents do not make a big mistake if taking into account the uniform distribution instead of the normal.

In order to keep expressions simple I consider the normal distribution with variance n. If there are n agents the distribution of the sum of the differences of their private valuations will typically have a variance of $(n-1)\sigma^2$. This simplification does not affect the generality of my results. Consider the squared error of the approximation:

$$SQE(a,n) = \int_{-a}^{a} \left(\frac{1}{\sqrt{2\pi n}} e^{-\frac{x^2}{2n}} - \frac{1}{2a} \right)^2 dx + \int_{-\infty}^{-a} \left(\frac{1}{\sqrt{2\pi n}} e^{-\frac{x^2}{2n}} \right)^2 dx + \int_{a}^{\infty} \left(\frac{1}{\sqrt{2\pi n}} e^{-\frac{x^2}{2n}} \right)^2 dx$$

One can show that the above expression can be written as $SQE(a,n) = \frac{1}{2\sqrt{\pi n}} - \frac{1}{a}\left[2\Phi_n(a) - \frac{3}{2}\right]$, where Φ_n denotes the cumulative distribution function of the normal distribution with zero mean and variance equal to n. As the parameters a and n increase the squared error decreases towards zero. That is, for any $\varepsilon > 0$ one can find $\delta > 0$ such that with any $a, n > \delta$ I have $SQE(a,n) < \varepsilon$. ■

Proof of Proposition 5. For simplicity let us consider agent 1 as playing *against* other n agents in the economy. If $d_i \sim iiF$ with expected value 0 and variance σ^2, once we suppose that F has uniformly bounded third moments, we get that $\sum_{i=2}^{n+1} y_j^i \to N(0; n\sigma^2)$ in the sense of distribution. Moreover Berry (1941) shows that the error term of this approximation in the neighborhood of zero is of order $n^{-\frac{1}{2}}$. Now let us consider the error in the approximation of the normal density with a constant. I study the following tolerance measure for the goodness of the approximation; i.e., the first order Taylor approximation of the normal density around 0:

$$Tol = \max_{d_1 \in [d_{\min}; d_{\max}]} |f(d_1) - P_l(d_1, 0)|, \text{ where}$$

$$f(d_1) = \frac{1}{\sigma\sqrt{2\pi n}} \exp\left(-\frac{d_1^2}{2n\sigma^2}\right) \text{ and } P_1(d_1, 0) = \frac{1}{\sigma\sqrt{2\pi n}}.$$

Similarly to the two-player case, as shown in Proposition 3, individual bids are bounded also in the n-player case, both from above and below. Take $d^* = \max\{|d_{\min}|, |d_{\max}|\}$. The maximization problem in the definition of the tolerance function is solved at d^*, therefore:

$$Tol = \frac{1}{\sigma\sqrt{2\pi n}} \left| \exp\left(-\frac{d^{*2}}{2n\sigma^2}\right) - 1 \right|.$$

Note that the inequality $\exp\left(-\frac{1}{2n\sigma^2}\right) - 1 < 0$ holds always in our examples. For this reason the tolerance function can be simplified and its properties in the limit can be written as:

$$Tol\left(n;\sigma^2\right)=\frac{1}{\sigma\sqrt{2\pi n}}\left[1-\exp\left(-\frac{1}{2n\sigma^2}\right)\right],$$
$$\lim_{n\to\infty}Tol\left(n;\sigma^2\right)=0,\ \lim_{\sigma^2\to\infty}Tol\left(n;\sigma^2\right)=0,\text{ and }\lim_{n\sigma^2\to\infty}Tol\left(n;\sigma^2\right)=0.$$

As for the rate of convergence, note that we have that $\lim_{n\to\infty}\sqrt{n}Tol\left(n;\sigma^2\right)=0$. Hence one can conclude that $Tol\left(n;\sigma^2\right)=o_p\left(n^{-\frac{1}{2}}\right)$. Altogether, in two steps, it has been shown that the error term in the approximation around zero of the density of a sum of centered iid random variables, that have uniformly bounded third moments, with a constant is of order $n^{-\frac{1}{2}}$. ■

Proof of Proposition 6. This result follows immediately from Lemma 4 and Proposition 4. ■

Proof of Proposition 7. The result in Proposition 4 relies on the fact that the variance of D can be any large whenever the number of participants is large enough. Naturally the large variance of D can be due to the large variance of every single d_i, too. ■

Proof of Lemma 5. Suppose that agent i experiences private valuations with a difference of $d_1=x_1^1-x_1^2$. Her bid for project 1 in the equilibrium can be computed according to the optimal bidding function and will be equal to $B\left(d_1\right)$. Due to the rules of the multibidding mechanism, in particular to the fact that bids must sum up to zero, with this her bid for project 2 is $-B\left(d_1\right)$. Now I can consider situations in which for player 1 it is more convenient to compute her bid for project 2 first, i.e. to take into account $d_1^*=x_1^2-x_1^1=-d_1$. Of course, equilibrium bids can not change with the above technicality, therefore $B^*\left(-d_1\right)=B^*\left(d_1^*\right)=-B\left(d_1\right)$. Since the support of f and f^* coincides both bidding functions, B and B^* are well-defined. ■

Proof of Proposition 8. Proposition 8 follows from Proposition 3 and Lemma 3. The expected utility player 1 has to maximize in the symmetric set-up can be written as

$$v_1\left[x_1^1,d_1,B\left(y_1\right)\right]=\int_{-y_1}^{x_H}\left[x_1^1-\frac{1}{2}B\left(y_1\right)+\frac{1}{2}B\left(d_2\right)\right]f\left(d_2\right)dd_2+$$
$$+\int_{x_L}^{-y_1}\left[x_1^2+\frac{1}{2}B\left(y_1\right)-\frac{1}{2}B\left(d_2\right)\right]f\left(d_2\right)dd_2.$$

In the next steps I shall transform the above expression in order to get (5) that will allow us to use the solution from Proposition 3. Now let us introduce the following change in the variables: $-d_2^*=d_2$. Note that since the support of f and f^* is the same I have that

$x_L = x_H^*$ and $x_H = x_L^*$.

$$v_1\left[x_1^1, d_1, B(y_1)\right] = -\int_{y_1}^{x_L}\left[x_1^1 - \frac{1}{2}B(y_1) + \frac{1}{2}B(-d_2^*)\right] f(-d_2^*)\, dd_2^* +$$
$$-\int_{x_M}^{y_1}\left[x_1^2 + \frac{1}{2}B(y_1) - \frac{1}{2}B(-d_2^*)\right] f(-d_2^*)\, dd_2^* =$$
$$= \int_{x_L}^{y_1}\left[x_1^1 - \frac{1}{2}B(y_1) - \frac{1}{2}B^*(d_2^*)\right] f^*(d_2^*)\, dd_2^* + \int_{y_1}^{x_M}\left[x_1^2 + \frac{1}{2}B(y_1) + \frac{1}{2}B^*(d_2^*)\right] f^*(d_2^*)\, dd_2^*.$$

If $B(\cdot)$ represents player 1's bid (bidding function) for project 1 in equilibrium, $B^*(\cdot)$ in the above expression can be interpreted as player 2's bid for the alternative project 2. The variables these functions depend on once again have the same distribution, i.e. I am back in the asymmetric case. Proposition 8 can be derived from (8) applying the solution from Proposition 3. ■

Proof of Proposition 9. From Proposition 3 and Proposition 8 we have that the optimal bidding function is symmetric, $B(-d_i) = -B(d_i)$ for every d_i and every i, if and only if the prior distribution is symmetric around 0. Let me show first that symmetry implies ex-post efficiency. To see that note that $x_1^1 + x_2^1 \geq x_2^2 + x_2^2 \rightarrow d_1 \geq -d_2$. By strict monotonicity of the bidding function $B(d_1) \geq B(-d_2)$, that implies ex-post efficiency if the symmetry condition holds:

$$B(d_1) \geq -B(d_2) \rightarrow B(d_1) + B(d_2) \geq 0.$$

For the reverse implication note first that by efficiency

$$\begin{aligned} d_1 + d_2 &= 0 \leftrightarrow B(d_1) + B(d_2) = 0, \\ -d_1 + d_1 &= 0 \leftrightarrow B(-d_1) + B(d_1) = 0, \end{aligned}$$

that gives the symmetry condition of $B(-d_1) = -B(d_1)$. ■

7.1 Simulation

This subsection contains theoretical results that have been used in the simulation process for the uniform, $U[-1;1]$, example with more than two bidders. Final results of the simulation are resumed in Table 1. in the main text. In order to analyze the general n-player as a special case with only two players, e.g. player 1 and the rest of the agents, the following pieces of notations are introduced: $D_{-1} = B^{-1}\left[\sum_{i=2}^{n} B(d_i)\right]$.

The distributions of the random variables in question are $d_i \sim iiF_d$ and $B(d_i) \sim iiF_{B(d)}$, and by the central limit theorem $\sum_{i=2}^{n} B(d_i) \overset{a}{\sim} N(\mu; \sigma^2)$. Now I can state a symmetry result on the optimal bidding function in the general n-player case.

Lemma 6 *If n is large, the distribution of D_{-1} is symmetric if and only if B is symmetric; i.e., $B(-d_i) = -B(d_i)$ for every d_i.*

Proof. Note that the following relations hold between distribution and density functions:

$$
\begin{aligned}
&\sum_{i=2}^{n} B(d_i) \sim F_\Sigma, f_\Sigma; \\
F_D(x) &= pr[D_{-1} \leq x] = pr\left[\sum_{i=2}^{n} B(d_i) \leq B(x)\right] = F_{B(d)}[B(x)] = F_\Sigma[B(x)]; \\
f_D(x) &= \frac{\partial F_D(x)}{\partial x} = \frac{\partial F_\Sigma[B(x)]}{\partial x} = f_\Sigma[B(x)] \cdot B'(x).
\end{aligned}
$$

Now let me consider the first implication in the proposition with the following equalities: $f_D(-x) = f_\Sigma[B(-x)] \cdot B'(-x)$. If the optimal bidding function B is symmetric we also have that $f_\Sigma[-B(x)] \cdot B'(x) = f_\Sigma[B(x)] \cdot B'(x) = f_D(x)$. That is the underlying distribution is symmetric.

In order to prove the proposition in the opposite direction, suppose that the distribution characterized by F_D is symmetric. Now one has that

$$
\begin{aligned}
F_D(-x) &= 1 - F_D(x); \\
F_\Sigma[B(-x)] &= 1 - F_\Sigma[B(x)]; \\
B(-x) &= -B(x).
\end{aligned}
$$

That is the optimal bidding function B is symmetric. ■

Even if I can not compute the optimal bidding function in the general case, I can deliver a mathematical expression for its explicit form that is useful in the simulation.

Proposition 10 *The optimal bidding function in the case of n bidders can be written as*

$$
B(d_1) = \left\{ \begin{array}{ll}
\frac{1}{2}d_1 + \frac{1}{2}[1-2F(d_1)]^{-\frac{n}{n-1}} \cdot \int_{d_1}^{d_M} [1-2F(t)]^{\frac{n}{n-1}} dt & \text{if } d_1 < d_M \\
\frac{d_1}{2} & \text{if } d_1 = d_M \\
\frac{1}{2}d_1 - \frac{1}{2}[2F(d_1)-1]^{-\frac{n}{n-1}} \cdot \int_{d_M}^{d_1} [2F(t)-1]^{\frac{n}{n-1}} dt & \text{if } d_1 > d_M
\end{array} \right\}
$$

where F is the cumulative distribution function of $D_{-1} = B^{-1}\left[\sum_{i=2}^{n} B(d_i)\right]$.

Proof. Let me define $D_{-1} = B^{-1}\left[\sum_{i=2}^{n} B(d_i)\right] \sim F, f$. Now project 1 wins if $B(d_1) + \sum_{i=2}^{n} B(d_i) \geq 0$, that is when $B^{-1}[-B(d_1)] \leq D_{-1}$. Since the distribution of D_{-1} is symmetric by the previous lemma we have that the optimal bidding function is also symmetric. With this, project 1 wins if $-d_1 \leq D_{-1}$.The expected utility for agent 1 can be written as

$$v_1\left[x_1^1, d_1, B\left(y_1\right)\right] = \int_{-d_1}^{x_H} \left\{x_1^1 - B\left(y_1\right) + \frac{1}{n}\left[B\left(y_1\right) + B\left(D_{-1}\right)\right]\right\} f\left(D_{-1}\right) dD_{-1} +$$
$$+ \int_{x_L}^{-d_1} \left\{x_1^2 + B\left(y_1\right) - \frac{1}{n}\left[B\left(y_1\right) + B\left(D_{-1}\right)\right]\right\} f\left(D_{-1}\right) dD_{-1}.$$

Agents are supposed to maximize their expected utility in the bidding. The first order condition of the problem is $\frac{\partial}{\partial y_1} v_1\left[x_1^1, d_1, B\left(y_1\right)\right] = 0$ that gives the following results: the optimal bidding function must solve the differential equation below for $y_1 = d_1$.

$$\left[\left(1-\frac{1}{n}\right) - 2\left(1-\frac{1}{n}\right) F\left(d_1\right)\right] B'\left(d_1\right) - 2B\left(d_1\right) f\left(d_1\right) + d_1 f\left(d_1\right) = 0$$

The solution for the differential equation is:

If $\left(1-\frac{1}{n}\right) - 2\left(1-\frac{1}{n}\right) F\left(d_1\right) = 0$; i.e., $F\left(d_1\right) = \frac{1}{2}$ then $B\left(d_1\right) = \frac{d_1}{2}$.

If $F\left(d_1\right) \neq \frac{1}{2}$ then

$$B'\left(d_1\right) - 2\frac{f\left(d_1\right)}{\left(1-\frac{1}{n}\right) - 2\left(1-\frac{1}{n}\right) F\left(d_1\right)} B\left(d_1\right) + d_1 \frac{f\left(d_1\right)}{\left(1-\frac{1}{n}\right) - 2\left(1-\frac{1}{n}\right) F\left(d_1\right)} = 0.$$

Let me introduce the notation $A\left(d_1\right) = \frac{f\left(d_1\right)}{\left(1-\frac{1}{n}\right) - 2\left(1-\frac{1}{n}\right) F\left(d_1\right)}$ and $x \in \left[x_L, x_H\right]$. The latter identifies the lowest and the largest admissible value for x. The differential equation and its general solution can be written now as

$$B'\left(d_1\right) - 2A\left(d_1\right) B\left(d_1\right) + d_1 A\left(d_1\right) = 0,$$
$$B\left(d_1\right) = \exp\left(2\int_x^{d_1} A\left(t\right) dt\right) \cdot \left\{\eta - \int_x^{d_1} \left[tA\left(t\right) \cdot \exp\left(-2\int_x^t A\left(s\right) ds\right)\right] dt\right\}.$$

Note that the integrals in the solution might include a difference such that $A\left(d_1\right)$ is not defined, therefore x must be carefully chosen. This parameter along with η can be fixed taking into account that the optimal bidding function must be continuous and strictly increasing.

For $d_1 > d_M$ fix some $x > d_M$ and choose η such that $B'\left(d_1\right) > 0$.

$$\int_{x_H}^{d_1} \left[tA\left(t\right) \cdot \exp\left(-2\int_{x_H}^t A\left(s\right) ds\right)\right] dt =$$
$$= -\frac{1}{2} d_1 \cdot \left[2F\left(d_1\right) - 1\right]^{\frac{n}{n-1}} + \frac{1}{2} x_H + \frac{1}{2}\int_{x_H}^{d_1} \left[2F\left(t\right) - 1\right]^{\frac{n}{n-1}} dt$$

$$B\left(d_1\right) = \frac{1}{2} d_1 + \left[2F\left(d_1\right) - 1\right]^{-\frac{n}{n-1}} \cdot \left[\eta - \frac{1}{2} x_H + \frac{1}{2}\int_{d_1}^{x_H} \left[2F\left(t\right) - 1\right]^{\frac{n}{n-1}} dt\right]$$

Let us choose $\eta = \frac{1}{2}x_H - \frac{1}{2}\int_{d_M}^{x_H} [2F(t) - 1]^{\frac{n}{n-1}} dt$.

One can solve similarly for the opposite case, $d_1 < d_M$. Finally one gets that

$$B(d_1) = \left\{ \begin{array}{ll} \frac{1}{2}d_1 + \frac{1}{2}[1 - 2F(d_1)]^{-\frac{n}{n-1}} \cdot \int_{d_1}^{d_M} [1 - 2F(t)]^{\frac{n}{n-1}} dt & \text{if } d_1 < d_M \\ \frac{d_1}{2} & \text{if } d_1 = d_M \\ \frac{1}{2}d_1 - \frac{1}{2}[2F(d_1) - 1]^{-\frac{n}{n-1}} \cdot \int_{d_M}^{d_1} [2F(t) - 1]^{\frac{n}{n-1}} dt & \text{if } d_1 > d_M \end{array} \right\}.$$

■

The problem with the above result is that the formula for $B(d_1)$ implicitly contains the inverse of the optimal bidding function, because the distribution function F is defined in $D_{-1} = B^{-1}\left[\sum_{i=2}^{n} B(d_i)\right] \sim F, f$. That is $\sum_{i=2}^{n} B(d_i) \overset{a}{\sim} N(\mu; \sigma^2)$ and $F_D(x) = F_\Sigma[B(x)]$. But we can use these results for simulating the optimal bidding function and computing a measure for its efficiency. The optimal bidding function is determined according to the following iterative procedure:

1. Take as given $\sum_{i=2}^{n} B(d_i) \sim F_\Sigma$, possibly some $N(\mu; \sigma^2)$, and compute with it $B_{F_1}(d_1)$.

2. Compute $F_2(d_1) = F_\Sigma[B_{F_1}(d_1)]$.

3. Using the resulting distribution function $F_2(d_1)$ from the previous point compute $B_{F_2}(d_1)$.

4. Repeat the procedure 1.-3. until the result converges, that is $\max_{d_1} \left|B_{F_{n-1}}(d_1) - B_{F_n}(d_1)\right| < \varepsilon$ for some predefined $\varepsilon > 0$.

In the example presented in the paper $\varepsilon = 10^{-5}$ and I have used 501 evaluation points in the $[-1; 1]$ interval in order to plot the optimal bidding function. The number of ex post efficient decisions has been approximated by a Monte Carlo experiment with 50000 draws. Results are presented in Table 1. in the main text of the paper.

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