The Forward Solution for Linear Rational Expectations Models *

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Abstract

This paper generalizes the forward method of recursive substitution and the corresponding forward solution to a general class of linear Rational Expectations models with predetermined variables. This forward method detects the existence of the fundamental solution to a given model by simply verifying whether the model can be solved forward and its solution does not depend on the expectations of the future endogenous variables, a property known as the no-bubble condition or transversality condition. The resulting forward solution is the relation between the endogenous and state variables implied by the recursive structure of the model. Consequently, the forward solution satisfies the no-bubble condition and it is unique in the class of fundamental solutions by construction whenever it exists, independent of determinacy of a given model. While there may exist other fundamental solutions, we show that they must violate the no-bubble condition despite being fundamental, bubble-free solutions. We provide several examples where seemingly legitimate fundamental solutions obtained by other methods may not be admissible as economically sensible Rational Expectations solutions.

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1 Introduction

It is well-known that Rational Expectations (RE) models can have multiple solutions. There can be an infinity of non-fundamental solutions and a multiple, but finite number of fundamental solutions. While the literature has developed a number of methods and strategies to obtain and analyze the different solutions, two important issues remain unresolved. First, when both fundamental and non-fundamental solutions coexist, which class of solutions is more relevant to a given model? Second, out of the multiple fundamental solutions, which is the most economically sensible one? This paper is an attempt to give an answer to this second question. To do so, we apply the traditional forward method of recursive substitution to a very general class of linear RE models.

In textbook-style univariate RE models without predetermined variables such as the Cagan (1956) or the Samuelson (1958) models, the state variables are simply the exogenous processes and it is thus standard to solve these models forward recursively. The forward method yields the model-implied forward representation, where the current endogenous variable is related to the expected future endogenous variable plus the current exogenous variables, whenever the stochastic processes of the exogenous variables are known. The forward (also known as forward-looking) solution is defined when such a relation is stable in the limit and the expected future endogenous variable does not affect the current endogenous variable. The latter condition is typically denoted as the no-bubble condition, the transversality condition, or, in some contexts, a boundary or terminal condition. As such, the forward solution can be interpreted as a model-implied relation between the endogenous variables and state variables, implying that it is a bubble-free fundamental solution. The forward solution concept is thus straightforward and, to our knowledge, there has been no counterargument against its validity as a RE solution.

However, in modern macroeconomic models such as the popular linear dynamic stochastic general equilibrium RE systems with lagged predetermined variables, the forward method and the forward solution have not been formally employed or analyzed. Instead, alternative solution techniques have been developed by Blanchard and Kahn (1980), Uhlig (1997), King and Watson (1998), McCallum (1999), Klein (2000) and Sims (2001), among others. These solution methods can examine the determinacy of a given model using eigenvalue-eigenvector decompositions, and yield exact, though typically not analytical, fundamental solutions.

The problem of multiplicity of stationary solutions can naturally arise in such models. As a result, researchers have proposed solution selection criteria to choose one economically meaningful solution to a given model. Among the best known criteria are the minimum state variable (MSV) criterion, designed by McCallum (1983)[1] and the E-Stability criterion of Evans and Honkapohja (2001), which is based on a learning process. Unfortunately, however, no consensus has been reached on the "right" solution to a given model. McCallum (2004) provides an example where the solution obtained via the MSV criterion does not coincide with the unique stationary solution to the model. The E-Stability criterion, in turn, needs a particular learning process and, as Evans and Honkapohja (2001) show, several solutions can pass the E-Stability criterion under adaptive learning.

In this paper we show that the forward method can be applied to the class of general multivariate linear RE models with predetermined variables. We also show that the resulting forward solution has isomorphic properties to the standard forward solution in univariate linear RE models without predetermined variables. Specifically, when a given model is solved forward, the current endogenous variable is related to the expected future endogenous variables and the state variables, which are now both predetermined

[1]The solution obtained via the MSV criterion is often called the MSV solution in the literature. However, the fundamental solutions are also called in general MSV solutions, because they depend on the minimal set of state variables. To avoid confusion throughout the paper, we restrict the term MSV solution to that obtained via the solution method proposed by McCallum, whereas fundamental solutions will denote the solutions that depend on the minimal set of state variables.

and exogenous variables. The forward solution is then simply the model-implied relation in the limit between the endogenous and the state variables in the absence of the effect of the expected future endogenous variables on the current endogenous variable. Therefore, the forward solution must satisfy the no-bubble condition (NBC hereafter) by definition. The forward solution belongs to the class of fundamental solutions and is unique by construction, if it exists. More importantly, if the forward solution exists, we show that it is the only one that satisfies the NBC. This is important because, independently of the determinacy of a given model, one can directly examine the existence of a bubble-free, fundamental solution, i.e. the forward solution. If this solution exists, it is straightforward to obtain it via the forward method.

The fact that only the forward solution satisfies the NBC has key implications for the remaining fundamental solutions in these two plausible scenarios: First, if the forward solution exists and there also exist other fundamental solutions, then the remaining solutions fail to pass the NBC despite being fundamental solutions. Second, it may be that the forward solution does not exist but there exist fundamental solutions. Then the expected future endogenous variables explode or oscillate as the recursion continues in the forward representation of the model, again violating the NBC. This is important because a seemingly valid fundamental solution, categorized as a bubble-free solution in either case, turns out to violate the no-bubble condition.

A key distinction between the forward and the existing methods is that whereas the former examines the model-implied relation between the endogenous and the state variables, the latter go in the opposite direction: They first characterize the set of solution candidates and then elucidate which one solves the model. Therefore, while multiplicity of fundamental solutions can naturally arise with the currently used methods, the forward method always delivers a unique fundamental solution (if it exists) by construction.

In addition to the technical difference just mentioned, the forward method is also an economically more sensible way to obtain the fundamental solution to a model. Given the recursive structure present in a RE model, the forward-looking agents deduce the relation between the endogenous and the state variables recursively in a forward-looking manner. In infinite-horizon RE models, it is natural that the expectations of the future endogenous variables very far into the future do not affect the dynamics of the current endogenous variables. This is precisely the idea of the standard forward solution and we show in this paper that it does apply to the class of general linear RE models. In contrast, it is hard to infer an economic intuition from the solution methods based on eigenvalue-eigenvector decomposition theories.

A closely related work is Binder and Pesaran (1997). They impose certain terminal conditions on the conditional expectations of the future endogenous variables and solve the model backward recursively. Driskill (2006) also proposes a similar method based on backward induction and illustrates his technique using several popular examples. His examples are confined to the univariate case and his method is not generalized. Our method is different from theirs in that we solve a general linear RE model forward without imposing a specific terminal condition. In contrast, they *assume* and impose a specific NBC, which is solution dependent. In our proposed forward-method, we instead *verify* that only the forward solution satisfies the NBC, whereas all the other solutions do not.

In addition to the set of fundamental solutions, there may exist an infinite number of sunspot solutions, as explained by Farmer and Guo (1994) and discussed by Lubik and Schorfheide (2004). These solutions must violate the NBC, as they are defined to do so. As mentioned above, when both classes of model solutions coexist, it is an open question which solution is more relevant to a given model. In this paper, we do not take a stand on this issue, as the scope of this paper is confined to the class of fundamental solutions.

The paper proceeds as follows. Section 2 reviews the key properties of the standard forward solution in a univariate RE model without predetermined variables. Section 3 explains our forward method and the forward solution using a simple univariate RE model with predetermined variables and compares it with other existing methods. We also provide a graphical analysis of the forward method. Section 4 generalizes the forward method and the forward solution to a general class of linear multivariate RE models. Section 5 provides several examples illustrating the differences between the forward solution and other solutions. In particular, we show that the solutions obtained through the existing methods can differ from the forward solution, implying that they violate the NBC. Section 6 concludes.

2 Univariate Models without Predetermined Variables

We start with a simple univariate linear Rational Expectations (RE) model in the absence of predetermined variables. Even though this is a well understood model, it is very instructive to do so for two reasons: First, it provides a very clear and intuitive benchmark for the subsequent discussion of the forward solution in more complex models. Second, it allows us to introduce the two key conditions needed to characterize the forward solution.

The univariate RE model without predetermined variables can be expressed as:

$$x_t = aE_t x_{t+1} + z_t \tag{1}$$

where x_t is an endogenous variable and E_t is the mathematical expectation operator conditional on information available at time t. z_t is an exogenous forcing variable, the only state variable in this model. The parameter a is unrestricted in order to encompass various popular models.[2] We assume that z_t follows a stationary AR(1) process:

$$z_t = \rho z_{t-1} + \epsilon_t, \tag{2}$$

$|\rho| < 1$ and ϵ_t is a white noise process. The class of the solutions discussed in this paper is the set of fundamental solutions where the endogenous variables exclusively depend on the minimal set of the state variables, which in this model is simply z_t:

$$x_t = \gamma z_t, \tag{3}$$

where $\gamma = 1/(1 - a\rho)$. The fundamental solution does not exist when $a\rho = 1$. It is important to note that the number of fundamental solutions here is finite and it is at most one, if it exists. In the case of indeterminacy, there exist an infinite number of non-fundamental bubble solutions, such as:

$$x_t = (1/a)(x_{t-1} - z_{t-1}) + w_t \tag{4}$$

where w_t is an arbitrary martingale process such that $E_{t-1} w_t = 0$.

We now briefly review the standard forward method of recursive substitution and the resulting forward solution. Solving the model forward using the law of iterative expectations is equivalent to a forward representation of the model as follows:

$$x_t = a^k E_t x_{t+k} + \gamma_k z_t \tag{5}$$

$$\gamma_k = \sum_{i=1}^{k} (a\rho)^{i-1} \tag{6}$$

[2] For instance, in the Cagan (1956) model, x_t is the (log) price level, z_t is the (log) nominal money stock and a is typically smaller than 1. But a can be greater than 1 in the overlapping generations model of Samuelson (1958). In the intertemporal IS equation, x_t and z_t are the log consumption and the expected (exogenous) real interest rate, respectively and $a = 1$.

for $k = 1, 2, ...$ Note that any solution, either fundamental or non-fundamental, must satisfy the forward representation (5), because it is implied by the model. Suppose that the coefficient of the state variable, γ_k converges, so that:

$$\gamma^* = \lim_{k\to\infty} \gamma_k = 1/(1 - a\rho). \tag{7}$$

This implies that the function of the state variable $\gamma^* z_t$ is stationary and independent of k. For future reference, this condition will be called the Forward Convergence Condition (FCC) of the coefficient on the state variables. The forward solution in the literature is defined as the model-implied forward representation in the limit, which is a function of the state variable only:

$$x_t = \gamma^* z_t, \tag{8}$$

by *assuming* the following No-Bubble Condition (NBC):[3]

$$\lim_{k\to\infty} a^k E_t x_{t+k} = 0 \tag{9}$$

We emphasize the following two crucial properties of the forward solution and we show that those principles apply to the models with predetermined variables: First, the forward solution exists if the FCC holds. The forward representation relates the current endogenous variable recursively with its own expected future value and the state variable for each k. Clearly, it is not be desirable for any model that such a relation be not stable. Since (5) must hold for any arbitrary k and arbitrary solution, the violation of

[3]There seems to be no single terminology to denote this condition in the literature. No-bubble condition is the most common one in asset pricing equations. In the context of fiscal policy, it is called the No-Ponzi Game Condition or the Intertemporal Budget Condition (see Walsh (2003)), or also the Terminal Condition (Devereux and Mansoorian (1992)). In alternative macroeconomic models it is also called the Transversality Condition (Romer (1996)) or a Boundary Condition (Driskill (2006)) to pin down a solution.

the FCC implies that the term denoting the expected endogenous variable, $a^k E_t x_{t+k}$, also known as bubble term, becomes unstable as k increases, independent of which solution, fundamental or non-fundamental, is used in the formation of expectations. Note that the FCC is a model property independent of a particular solution. Consequently, the FCC is the minimal requirement necessary for a model to be well-specified and for any solution to be economically meaningful. The forward solution is the one defined to capture this desirable model property. In this model, the FCC holds when $|a\rho| < 1$ and the solution is precisely the standard "forward" solution in Blanchard (1979)'s sense.[4]

Second, the forward solution must satisfy the NBC and it is the only one which does so among *all* the solutions. Unlike the FCC, the NBC depends on a particular solution when forming expectations. Therefore, the NBC is a condition that must be verified with a particular solution in consideration, not assumed. It is straightforward to show that the forward solution satisfies the NBC; otherwise, it is a violation to the fact that it is the forward solution.[5] Hence, the standard NBC assumption may implicitly mean that one is searching for the forward solution. Any other solution different from the forward solution cannot satisfy the NBC. In particular, when the FCC does not hold, then none of the solutions obtained by other methods satisfies the NBC. Consequently, the FCC is not just a property for a model to be well-specified but it is also a sufficient condition for the existence of a unique fundamental solution that is bubble free, which is the forward solution.

The main message of this paper is derived from the existence of these two key properties: the FCC and the NBC. The forward solution is the unique fundamental solution that

[4]Blanchard (1979) shows that even when $|a| > 1$, if z_t is expected to return to its mean (here 0) fast enough, the forward solution is stationary. In our example, this amounts to $\lim_{k\to\infty} a^k E_t z_{t+k} = \lim_{k\to\infty} (a\rho)^k z_t = 0$.

[5]We can directly prove this. The present model satisfies the FCC if $|a\rho| < 1$. Then, the NBC with expectations formed with the forward solution becomes $\lim_{k\to\infty} a^k E_t x_{t+k} = (a\rho)^k \gamma^* z_t = 0$.

satisfies the NBC by construction, if it exists. Therefore, all the fundamental, bubble-free solutions other than the forward solution do not satisfy the NBC. Consider the case when the forward solution does not exist. In this model, it amounts to $|a\rho| \geq 1$. Suppose that one obtains a fundamental solution through other method, for example, the method of undetermined coefficients, $x_t = \gamma z_t$ where $\gamma = 1/(1 - a\rho)$. Thus, except for the case $a\rho = 1$, such a solution exists. Since z_t is assumed to be a bounded process, the solution $x_t = \gamma z_t$ is clearly a stationary fundamental solution. But this seemingly relevant solution is dismissed as a valid solution because $\lim_{k\to\infty} a^k E_t x_{t+k} = \lim_{k\to\infty}(a\rho)^k \gamma z_t$ explodes when $|a\rho| > 1$ or oscillates when $a\rho = -1$, violating the NBC.[6]

If the forward solution exists, then there is no other distinctive fundamental solution to this model. However, in models with predetermined variables there may well exist other fundamental solutions. Do such solutions violate the NBC? In the next section we show that the answer is yes and that this is the principle upon which we can discard alternative fundamental solutions.

3 Univariate Models with Predetermined Variables

Perhaps surprisingly and to our knowledge, the forward representation of a model with predetermined variables has not been formally developed and, consequently, the two key conditions aforementioned have not been examined in this context. In this section we derive the forward representation of a given model, the FCC, the NBC and the resulting forward solution in a univariate framework. We then provide a graphical analysis of our method. The essential features of the forward method and of the forward solution can

[6]For all non-fundamental solutions, the bubble term in (5) yields the same answer: $a^k E_t x_{t+k} = x_t - \gamma_k z_t \neq 0$ for all k and given a non-zero z_t, making (5) an identity. However, these solutions would make sense only when the model satisfies the FCC because otherwise, $a^k E_t x_{t+k}$ would explode. It is an important but unresolved question which class of solutions -fundamental or non-fundamental- is more relevant to a given model and period, but we do not discuss this issue in the paper.

be understood in the univariate framework, and they are generalized to the multivariate context in section 4.

Consider a simple univariate model with a predetermined variable:

$$x_t = aE_t x_{t+1} + bx_{t-1} + z_t \tag{10}$$

$$z_t = \rho z_{t-1} + \epsilon_t \tag{11}$$

where x_t is a univariate endogenous variable observed at time t and z_t is an exogenous variable. Here, the state variables are the predetermined variable, x_{t-1} and z_t. ϵ_t is assumed to be a white noise process.

It is instructive to distinguish in advance between the class of fundamental and non-fundamental solutions, even if the forward method does not require to do so. The fundamental solution to the model is given by:

$$x_t = \omega x_{t-1} + \gamma z_t \tag{12}$$

where (ω, γ) must belong to the following set of solution candidates:

$$\mathcal{A}=\{(\omega,\gamma)|\ \omega = (1-a\omega)^{-1}b, \quad \gamma = (1-a\omega)^{-1}(1+a\gamma\rho), \ (\omega,\gamma) \in \mathcal{R} \times \mathcal{R}\} \tag{13}$$

provided that $1-a\omega \neq 0$. $\mathcal{A}$ is the exhaustive and finite set of (ω, γ) consistent with (12) and therefore, the model has at most two solution candidates. Alternatively, the class of non-fundamental, bubble solutions is of the form:

$$x_t = \frac{1}{a}(x_{t-1} - bx_{t-2} - z_{t-1}) + w_t, \tag{14}$$

where w_t is an arbitrary martingale process.

3.1 The Forward Method and the Forward Solution

As in the previous model, we first present the forward representation of the model, followed by the definition of the FCC and NBC. Then we define the forward solution and examine its properties.

3.1.1 Forward Representation

Rewrite the model (10) with $m_1 = a$, $\omega_1 = b$, and $\gamma_1 = 1$, such that $x_t = m_1 E_t x_{t+1} + \omega_1 x_{t-1} + \gamma_1 z_t$. Shifting this equation forward one period and taking conditional expectations yields $E_t x_{t+1} = m_1 E_t x_{t+2} + w_1 x_t + \rho\gamma_1$, which depends on x_t. Replacing $E_t x_{t+1}$ and rearranging the model (10), we can derive $x_t = m_2 E_t x_{t+2} + \omega_2 x_{t-1} + \gamma_2 z_t$ where $m_2 = (1 - a\omega_1)^{-1} a m_1$, $\omega_2 = (1 - a\omega_1)^{-1} b$ and $\gamma_2 = (1 - a\omega_1)^{-1}(1 + a\gamma_1\rho)$. In this way, we can construct the unique set of sequences, $\{m_k, \omega_k, \gamma_k\}$ recursively as functions of the structural parameters, a and b, such that:

$$x_t = m_k E_t x_{t+k} + \omega_k x_{t-1} + \gamma_k z_t \tag{15}$$

where $m_1 = a$, $\omega_1 = b$, and $\gamma_1 = 1$, and for all $k = 2, 3, 4, ...$:

$$m_k = (1 - a\omega_{k-1})^{-1} a m_{k-1} \tag{16}$$

$$\omega_k = (1 - a\omega_{k-1})^{-1} b \tag{17}$$

$$\gamma_k = (1 - a\omega_{k-1})^{-1}(1 + a\rho\gamma_{k-1}) \tag{18}$$

The first thing we note in this forward representation is the similarity of the sequence (ω_k, γ_k) in (17) and (18) with the conditions in the set of solutions, $\mathcal{A}$. When (ω_k, γ_k) converge, (17) and (18) become the conditions of $\mathcal{A}$.

The only necessary condition for the existence of the forward representation is:

$$1 - a\omega_k \neq 0 \tag{19}$$

for all $k = 1, 2, 3, ...$ For ease of exposition, this condition is called the regularity condition. It is easy to see that ω and γ are real-valued when $ab \leq 1/4$. In this case, we show below that (19) is always satisfied.

3.1.2 FCC, NBC and the Forward Solution

We now formally define the forward convergence of the coefficients of the state variables, and the no-bubble condition. The model (10) is said to satisfy the Forward Convergence Condition (FCC) if the sequence (ω_k, γ_k) converges to (ω^*, γ^*) in the forward representation of the model. Under the FCC, the model implies:

$$x_t = \lim_{k\to\infty} m_k E_t x_{t+k} + \omega^* x_{t-1} + \gamma^* z_t. \tag{20}$$

A crucial part of the FCC is that $(\omega^*, \gamma^*) \in \mathcal{A}$ because equations (17) and (18) fulfill the conditions in $\mathcal{A}$. Under the FCC, $\lim_{k\to\infty} m_k E_t x_{t+k}$ is finite and invariant to k, independently of which solution, either fundamental or non-fundamental, is used in expectations formation. Next we define the No-Bubble Condition (NBC) of the model:

$$\lim_{k\to\infty} m_k E_t x_{t+k} = 0 \tag{21}$$

Note that the NBC can only hold if the FCC holds, and is solution-dependent. Consequently, if the model does not satisfy the FCC, any solution, fundamental or non-fundamental, obtained by other methods and other criteria, must violate the NBC.

The forward solution is thus defined as the model-implied forward representation of the model in the limit where the endogenous variable is a function of the state variable only:

$$x_t = \omega^* x_{t-1} + \gamma^* z_t \tag{22}$$

Therefore, the forward solution satisfies the NBC by construction and all other solutions violate the NBC because otherwise, it is a contradiction to the fact that the other solutions are different from the forward solution. We formally state this fact in the following Proposition.

Proposition 1: *Consider the model (10) together with (11):*

1. In the case that the forward convergence condition is satisfied, the forward solution defined as (22) is the unique real-valued fundamental solution to the model that satisfies the NBC.

2. For any other solution, either fundamental or non-fundamental, the NBC is violated, independently of the FCC.

Proof. *See Appendix A.* ■

Proposition 1 states that the forward solution is the only fundamental solution that is truly bubble free. In the forward representation of the model, one should expect that the expected endogenous variable far into the future does not affect the dynamics of the current endogenous variable when expectations are formed with a fundamental bubble-free solution. In this sense, the forward solution is the economically sensible one in the class of fundamental solutions. In contrast, the other fundamental solutions, which are assumed to be bubble free by construction, exhibit a bubble term which survives the forward recursion process of the structural model.

Notice that we do not require *a priori* information about the stationarity of the solution. If the forward solution exists, we conclude that it is stationary if $|\omega^*| < 1$. Note also that we do not need to verify the NBC because Proposition 1 states that once that the forward solution exists, it must satisfy the NBC. Therefore, technically speaking, one has only to compute the sequence of (ω_k, γ_k) in equations (17) and (18) from the structural parameters, a and b, and check the stability condition $|\omega^*| < 1$, in order to examine the existence of the stationary forward solution and obtain it, whenever it exists.

In the following subsection, we discuss the differences between the forward method and the forward solution and the other methods and fundamental solutions.

3.2 Relation With Other Solutions and Solution Methods

The forward method starts from a given model and examines a model-implied relation whereas alternative methods first characterize the solution candidates and then examine the different solutions to the model. Consequently, and in contrast to the forward method, the problem of multiple stationary solutions can arise in these other methods and a particular solution refinement scheme or selection criterion is needed, such as the E-stability criterion of Evans and Honkapohja (2001) or the MSV criterion of McCallum (1983), in order to choose one solution.

The existing solution procedures solve for the pair $(\omega, \gamma) \in \mathcal{A}$ by using, for instance, the method of undetermined coefficients. In our example, there are two fundamental solutions $x_t = \omega(s)x_{t-1} + \gamma(s)\epsilon_t$, $s = 1, 2$, where $(\omega(s), \gamma(s)) \in \mathcal{A}$ and $\omega(s)$ is a root of $a\omega^2 - \omega + b = 0$. Once $\omega(s)$ is determined, $\gamma(s)$ is determined. Suppose, without loss of generality, that $|\omega(1)| < |\omega(2)|$. Then we have the following results.

Corollary 1: *1. Suppose that the FCC holds. Then, $\mathcal{A}$ is non-empty, and $(\omega^*, \gamma^*) = (\omega(1), \gamma(1))$, i.e., the forward solution corresponds to the smallest root of ω. If the other solution exists $(\omega(2), \gamma(2))$, then $\lim_{k\to\infty} m_k E_t x_{t+k} = l_x(2)x_t + l_z(2)z_t \neq 0$ where* $l_x(2) = \lim_{k\to\infty} m_k \omega(2)^k \neq 0$ and $l_z(2) = \lim_{k\to\infty} m_k \sum_{i=1}^{k} \omega(2)^{k-i}\gamma(2)\rho^i$.

2. Suppose that the FCC does not hold. Then $\mathcal{A}$ may or may not be empty. If it is *not empty, the bubble term,* $\lim_{k\to\infty} m_k E_t x_{t+k}$*, either explodes or oscillates when expectations are formed with any one of the fundamental solutions.*

Proof. *See Appendix B.* ■

Corollary 1 states that as long as the forward solution exists, the results apply for all three cases where $|\omega(1)| < |\omega(2)| \leq 1$, $|\omega(1)| < 1 < |\omega(2)|$, $1 \leq |\omega(1)| < |\omega(2)|$, corresponding to the cases in which the model has multiple, unique and no stationary fundamental solutions, respectively. Therefore, the forward method does not need to characterize determinacy of the model. Suppose that one obtains an alternative solution through other solution method and selection device.[7] Then the bubble term $\lim_{k\to\infty} m_k E_t x_{t+k}$ under this solution converges, but not to zero: $\lim_{k\to\infty} m_k E_t x_{t+k} = l_x(2)x_t + l_z(2)z_t$ depends on the current endogenous variable, x_t and potentially on the current exogenous variable z_t.[8,9]

The second part of Corollary 1 might be more critical in practice. Suppose that the FCC does not hold. Using standard solution techniques, one may choose the solution $(\omega(1), \gamma(1))$ or the other one as a valid solution to the model. But since the FCC does not hold, the bubble term explodes when expectations are formed with a seemingly relevant solution. In a model without a predetermined variable, this case occurs when $|a\rho| > 1$. In that case, it is straightforward to detect that there is no bounded forward solution when the forward method is used. However, as we show below, in more complex models one may mistakenly choose a model solution where the FCC fails to hold.

[7] In this simple model, most of the existing solution selection criteria would choose $(\omega(1), \gamma(1))$ as a valid solution to the model. However, in section 4 we provide several multivariate examples where these selection criteria pick up different solutions from the forward solution, thus violating the NBC.

[8] This fundamental solution, $x_t = \omega(2)x_{t-1} + \gamma(2)z_t$ is still a bubble-free solution in the sense that it depends only on the minimum state variables. Therefore, a violation of the NBC should be understood as implying that the expected endogenous variables far into the future affect the current endogenous variables. In this instance, the bubble term survives even when such a fundamental solution is used in expectation formation. This is a phenomenon which arises in models with predetermined variables.

[9] One may argue that the non-zero bubble term $l_x(2)x_t + l_z(2)z_t$ is a different terminal condition, and thus should not rule out this solution. However, such a terminal condition that depends on current variables is hard to justify economically.

3.3 Graphical Representation of the Forward Method

In this subsection, we show how the forward solution can be described graphically. For simplicity we assume that $\rho = 0$. Recall that in the univariate case, for a solution to be real-valued, it must be the case that $\theta \equiv ab \leq 1/4$. Let $v_k = (1 - a\omega_k)$ be the sequence in equation (19). Then $v_1 = 1 - \theta$. Starting from v_1, it can be shown that for $k = 1, 2, 3, ...$,

$$v_{k+1} = 1 - \theta / v_k \tag{23}$$

Let $v(1) = \frac{1+\sqrt{1-4\theta}}{2}$ and $v(2) = \frac{1-\sqrt{1-4\theta}}{2}$ be the solutions of $v = 1 - \theta/v$. Then it is easy to see that $\omega(s) = v(s)^{-1}b$ solves the condition in $\mathcal{A}$ for $s = 1, 2$.

For the range of $0 < \theta \leq 1/4$, $v_1 \geq v(1) \geq 1/2$, Panel A of Figure 1 shows that v_k is well defined for all $k = 1, 2, ...$ and that it converges monotonically to $v(1) \geq 1/2$.[10] Panel B shows that for the range of $\theta < 0$, v_k is also well defined and it converges to $v(1) > 1$ with oscillation. In both cases, $v_k \geq 1/2 > 0$ for all $k = 1, 2, 3, ...$[11] Therefore, the regularity condition is not binding and the FCC holds for ω_k.

Figure 1 also shows graphically how the forward solution can be obtained. Starting from v_1, v_k converges to $v(1)$. This implies that ω_k converges to ω^*, where $\omega^* = (1 - a\omega^*)^{-1}b = v(1)^{-1}b$.[12] We emphasize that the initial value of v_1 is given by the model parameters as $1-\theta$, not as an arbitrary value. This is the reason why the forward solution

[10] The case of $\theta = 0$ becomes trivial. This can happen when $a = 0$ or $b = 0$. If $a = 0$, the model becomes purely backward-looking model, whereas if $b = 0$, it is purely forward-looking. In both cases, $v_k = 1$ and $\omega_k = b$ for all k.

[11] Suppose that $v_k \geq 1/2$. Then $-\theta/v_k \geq -2\theta \geq -1/2$ for all $\theta \leq 1/4$. Therefore, $v_{k+1} = 1 - \theta/v_k \geq 1/2$. Since $v_1 = 1 - \theta \geq 1/2$, and $v_k = 1 - \theta/v_{k-1}$ for all $k \geq 2$, $v_k = 1 - a\omega_k \geq 1/2$ for all $k \geq 1$.

[12] The forward method can also be applied with repeated eigenvalues, i.e., when $\theta = 1/4$. In this instance, equation (23) and $v_{k+1} = v_k$ are tangent at $v = 1/2$ and v_k converges to $1/2$ from $v_1 = 3/4$. However, since the slope at $v = 1/2$ is 1, we conjecture that the speed of convergence would be much lower. Indeed, the convergence speed of the forward solution is faster the more distant the two roots of v (or ω in $\mathcal{A}$) are, because the slope of v_k in Panel A is flatter at $v(1)$. When $a = 0.75$ and $b = 1/3$, $\omega(1) = \omega(2) = 2/3$. In this case, more than 1000 recursions are needed to attain a precision to the third decimal point. When $a = 0.749$, $\omega(1) = 0.643$, $\omega(2) = 0.692$, it takes only 57 recursions to reach the same precision.

is always unique if it exists. Furthermore, it is not necessary to solve for the roots of v (or ω). In contrast, standard solution methods essentially characterize the two roots of v and the corresponding two solutions of $\omega = v^{-1}b$, and select one solution through a particular selection device if both roots of ω are less than unity in absolute value.

In this model, the FCC is violated only if $\theta > 1/4$. Figure 2 illustrates the path of v_k when $\theta > 1/4$. Panel A shows that as long as the regularity condition is not violated, v_k oscillates, implying that ω_k does not converge. Panel B illustrates the case where the regularity condition is violated. Suppose that the regularity condition is violated at $k = K \geq 1$, i.e., $v_k \neq 0$ for $k = 1, 2, ..., K-1$ and $v_K = 0$.[13] Note that $\omega_k = v_{k-1}^{-1}b$ from equation (17). This implies that ω_1 through ω_K are well defined, whereas ω_{K+1} and v_{K+1} cannot be defined. We may interpret that v_{K+1} jumps to infinity (or minus infinity) and correspondingly $\omega_{K+1} = -\infty$ (∞). Then, from (23), $v_{K+2} = 1$ and $\omega_{K+2} = 0$. Finally $v_{K+3} = v_1$ and $\omega_{K+3} = \omega_1 = b$. That is, when the regularity condition is violated at $k = K$, the patterns of $\{\omega_k\}_{k=1}^{K+2}$ are repeated periodically. This implies that when the regularity condition is violated, ω_k does not converge and the forward solution does not exist.

4 Multivariate Linear Rational Expectations Models

In this section we generalize the forward method and the forward solution to a general linear multivariate RE model. The forward method and the forward solution are essentially identical to those in the univariate models.

[13] In this case, the violation of the regularity condition implies that the model becomes:

$$0 = (1 - a\omega_K)x_t = m_K E_t x_{t+K+1} + \omega_K x_{t-1} + \gamma_K \epsilon_t$$

Therefore, economic agents cannot relate the current variable to the K-th and higher order forward-looking terms recursively, even if they have been able to do so up to the $(K-1)$-th order. An implication of this point is that the regularity condition is a property that a well defined RE model needs to satisfy.

Consider the following standard model:

$$B_1 x_t = \alpha_0 + A_1 E_t x_{t+1} + B_2 x_{t-1} + C_1 z_t \tag{24}$$

where x_t is an $n \times 1$ vector of endogenous variables, α_0 is an $n \times 1$ vector of constants and B_1, A_1 and B_2 are $n \times n$ coefficient matrices of structural parameters. We assume that B_1 is a non-singular matrix but A_1 and B_2 can be singular. z_t is an $m \times 1$ vector of exogenous variables whose data generating process is known. C_1 is an $n \times m$ coefficient matrix of z_t. The information set available at time t includes all the current and past endogenous and exogenous variables. Pre-multiplying both sides by B_1^{-1} and assuming that z_t follows a VAR(1) law of motion, the model can be represented as:

$$x_t = \alpha + A E_t x_{t+1} + B x_{t-1} + C z_t \tag{25}$$

$$z_t = F z_{t-1} + \epsilon_t, \quad \epsilon_t \sim (0_{m\times 1}, D) \tag{26}$$

where $\alpha = B^{-1}\alpha_0$, $A = B_1^{-1}A_1$, $B = B_1^{-1}B_2$ and $C = B_1^{-1}C_1$. F is an $m \times m$ coefficient matrix and D is the $m \times m$ diagonal variance-covariance matrix of the residual vector z_t. $0_{m\times 1}$ denotes an $m \times 1$ matrix of zeros. The eigenvalues of F are assumed to be inside the unit circle. As Binder and Pesaran (1997) show, this model is quite general in the sense that it nests models with an arbitrary number of leads in the forward-looking variables, an arbitrary number of lags in the predetermined variables, and an arbitrary time at which expectations are formed.

We again present in advance the class of fundamental solutions although the forward method does not require this information. The class of fundamental RE solutions is of the following reduced-form:

$$x_t = c + \Omega x_{t-1} + \Gamma z_t \tag{27}$$

where c, Ω and Γ are $n \times 1$, $n \times n$ and $n \times m$ matrices, respectively. The complete set of real-valued solutions for c, Ω and Γ is given by:

$$\mathcal{A}=\{(c,\Omega,\Gamma)|\ (c,\Omega,\Gamma) \in \mathcal{R}^{n\times 1} \times \mathcal{R}^{n\times n} \times \mathcal{R}^{n\times m}\} \tag{28}$$

where c, Ω and Γ solve the following equations:

$$c = (I_n - A\Omega)^{-1}(\alpha + Ac) \tag{29}$$

$$\Omega = (I_n - A\Omega)^{-1}B \tag{30}$$

$$\Gamma = (I_n - A\Omega)^{-1}(C + A\Gamma F), \tag{31}$$

provided that $|I_n - A\Omega| \neq 0$ where I_n denotes an identity matrix of order n. There are at most ${}_{2n}C_n$ elements of $\mathcal{A}$.[14] We can rewrite equation (30) as:

$$A\Omega^2 - \Omega + B = 0. \tag{32}$$

It is now standard to solve this matrix quadratic form through the QZ method (see, for instance, Uhlig (1997), McCallum (1999), Klein (2000) and Sims (2001)). The QZ method can easily characterize the set of solution candidates, $\mathcal{A}$, with the generalized eigenvalues implied by the matrices of structural parameters A and B.[15] By inspecting the generalized eigenvalues, one can easily detect whether there is a unique or a multiple number of real-valued stationary fundamental solutions (see, for instance, Theorem 3 of Uhlig (1997)).

[14]The class of non-fundamental solution can also be described as $x_t = c + \Omega x_{t-1} + \Gamma z_t + b_t$ where b_t is an arbitrary process satisfying $AE_t b_{t+1} = (I - A\Omega)b_t$.

[15]Following Klein (2000), we define the generalized eigenvalues as the elements of the set $\lambda(\mathcal{N},\mathcal{M}) = v \in \mathcal{C} : |\mathcal{N} - v\mathcal{M}| = 0$, where $\mathcal{M} = \begin{bmatrix} A & 0_{n\times n} \\ 0_{n\times n} & I_n \end{bmatrix}$ and $\mathcal{N} = \begin{bmatrix} I_n & -B \\ I_n & 0_{n\times n} \end{bmatrix}$.

4.1 The Forward Method and the Forward Solution

We now explain in detail our forward method for multivariate RE systems.

4.1.1 Forward Representation

We first show that the model can be solved forward under a regularity condition. The forward representation of the model can be derived as follows.

Claim: *Consider equations* (25) *and* (26). *Suppose that* $\alpha,\ A, B, C, F$ *are real-valued. Then, there exists a unique sequence of real-valued matrix* $\{M_k, c_k, \Omega_k, \Gamma_k, k = 1, 2, 3, ...\}$ *such that:*

$$x_t = M_k E_t x_{t+k} + c_k + \Omega_k x_{t-1} + \Gamma_k z_t \tag{33}$$

where $M_1 = A$, $c_1 = \alpha$, $\Omega_1 = B$, $\Gamma_1 = C$, *and for* $k = 2, 3, ...,$

$$\begin{aligned} M_k &= (I_n - A\Omega_{k-1})^{-1} A M_{k-1} && (34) \\ c_k &= (I_n - A\Omega_{k-1})^{-1}(\alpha + A c_{k-1}) && (35) \\ \Omega_k &= (I_n - A\Omega_{k-1})^{-1} B && (36) \\ \Gamma_k &= (I_n - A\Omega_{k-1})^{-1}(C + A\Gamma_{k-1} F) && (37) \end{aligned}$$

if the following regularity condition is satisfied for all $k = 1, 2, 3, ...$:

$$|I_n - A\Omega_k| \neq 0 \tag{38}$$

Proof. *See Appendix C.* ■

Notice the similarity between (35), (36), (37) and (29), (30), (31), respectively. When these sequences converge, then their limits must be a member of $\mathcal{A}$.

The regularity condition is then the requirement under which a given RE model can be solved forward recursively for all leads. King and Watson (1998) provide a necessary condition for the existence of solutions which is equivalent to $|I_n - A\Omega| \neq 0_{n\times n}$ in our model. Note that this is the limiting case of our regularity condition, i.e. $|I_n - A\Omega^*| \neq 0_{n\times n}$. Our condition is stronger than theirs in that it requires that $|I_n - A\Omega_k| \neq 0_{n\times n}$ for all k. However, the regularity condition does not need to hold for other $\Omega \neq \Omega^*$ in $\mathcal{A}$. This condition is similar to that of Binder and Pesaran (1997) up to a finite k.

We have shown that in a univariate model, the regularity condition must hold if the parameters are restricted to guarantee the existence of real-valued solutions. Unfortunately, it is difficult to verify whether the regularity condition holds when the parameters are restricted to guarantee the existence of real-valued fundamental solutions in multivariate models. This difficulty arises because it is not possible in general to derive the conditions under which the solution is real-valued as an explicit function of the model parameters. It is however clear that when the regularity condition is violated, even if a real-valued stationary solution is obtained by other methods, it violates the FCC.

4.1.2 FCC, NBC and the Forward Solution

The FCC and the NBC are completely analogous to those in the univariate model. The model (25) is said to satisfy the Forward Convergence Condition (FCC) if the sequence $(c_k, \Omega_k, \Gamma_k)$ converge to $(c^*, \Omega^*, \Gamma^*)$ in the forward representation of the model. Under the FCC, the model implies:

$$x_t = \lim_{k\to\infty} M_k E_t x_{t+k} + c^* + \Omega^* x_{t-1} + \Gamma^* z_t, \tag{39}$$

and the No-Bubble Condition (NBC) of the model is given by:

$$\lim_{k\to\infty} M_k E_t x_{t+k} = 0 \tag{40}$$

The forward solution is defined as the model-implied forward representation of the model in the limit:

$$x_t = c^* + \Omega^* x_{t-1} + \Gamma^* z_t \tag{41}$$

All the key features of the univariate models are preserved: $(c^*, \Omega^*, \Gamma^*) \in \mathcal{A}$ is unique and real-valued, and the forward solution is the only one that satisfies the NBC. The following proposition is a generalization of Proposition 1.

Proposition 2: *Consider the model (25):*

1. If the FCC is satisfied, the forward solution defined as (41) is the unique real-valued fundamental solution to the model that satisfies the NBC.

2. For any other solution, either fundamental or non-fundamental, the NBC is violated, independently of the FCC.

Proof. *See Appendix D.* ■

The forward method is straightforward to implement. One simply constructs sequences of matrices from the model coefficients $(c_k, \Omega_k, \Gamma_k)$, checks the FCC and finally examines the eigenvalues of Ω^* to determine stationarity. In particular, the method can identify the source of the problem if there is no forward solution to a given model. For instance, it may be the case that the regularity condition is violated, or that only a subset of elements of Ω_k does not converge, or that Ω_k converges but Γ_k does not.

Proposition 2 makes clear that our forward method is not just a selection criterion applied only in the case of multiple fundamental solutions, but a complete procedure for solving RE models within the class of the fundamental solutions. It simultaneously detects the existence (or non-existence) of a fundamental solution that satisfies the NBC and provides a unique solution by construction.

4.2 Relation with Other Solutions

In order to compare our method to the existing ones, we further investigate the relation between the forward solution and the other solutions belonging to $\mathcal{A}$. Suppose that there are S real-valued solutions, $(c(s), \Omega(s), \Gamma(s))$ in $\mathcal{A}$ for $s = 1, 2, ..., S$:

$$x_t = c(s) + \Omega(s)x_{t-1} + \Gamma(s)z_t. \tag{42}$$

Without loss of generality, let $(c^*, \Omega^*, \Gamma^*) = (c(1), \Omega(1), \Gamma(1))$, if the forward solution exists. The following Corollary states that the NBC holds only for the forward solution.

Corollary 2. *1. Suppose that the FCC holds. Then, $\mathcal{A}$ is non-empty. If there exist other solutions, $(c(s), \Omega(s), \Gamma(s))$ for $2 \leq s \leq S$ and the expectations are formed with any one of those solutions, then $\lim_{k\to\infty} M_k E_t x_{t+k} = L_c(s) + L_x(s)x_t + L_z(s)z_t \neq 0_{n\times 1}$ where $L_x(s) = \lim_{k\to\infty} M_k \Omega(s)^k \neq 0_{n\times n}$, $L_c(s) = \lim_{k\to\infty} M_k \sum_{i=1}^{k} \Omega(s)^{i-1}c(s)$* and *$L_z(s) = \lim_{k\to\infty} M_k \sum_{i=1}^{k} \Omega(s)^{k-i}\Gamma(s)F^i$.*

2. Suppose that the FCC does not hold. Then $\mathcal{A}$ may or may not be empty. If it is not empty, the bubble term, $\lim_{k\to\infty} M_k E_t x_{t+k}$ does not converge when expectations are formed with other fundamental solutions.

Proof. *See Appendix E.* ■

If the forward solution exists, the first part of the Corollary states that when the expectations are formed with other fundamental solution $(c(s), \Omega(s), \Gamma(s))$, the coefficient matrices converge as k goes to infinity, but $L_x(s) \neq 0_{n\times n}$, because otherwise, it is a contradiction to $\Omega(s) \neq \Omega^*$; hence, the expectational term depends on the current endogenous variables. If the forward solution does not exist, $\lim_{k\to\infty} M_k E_t x_{t+k}$ does not converge for any s and therefore, the bubble term, $M_k E_t x_{t+k}$ depends on k as well as t.

As in the univariate case, the existing solution methods and selection criteria can fail to identify the fundamental, bubble-free solution for two reasons, which are related to the non-examination of the FCC and NBC. First, if the FCC fails to hold, any fundamental solution contains a bubble term. Second, a particular solution selection criterion may choose a fundamental solution that is not the forward solution, thus violating the NBC even under the FCC. In the following section, we provide several examples where these two problems arise and the forward solution detects them.

5 Illustrative Examples

In this section, we provide five numerical examples. The first three examples are based on standard New-Keynesian model where analytical solutions exist. Example 1 considers a case with a unique stationary fundamental solution. Example 2 illustrates a case where multiple stationary fundamental solutions exist and one of them is the forward solution. Example 3 is a case where the FCC fails to hold but the existing solution methods pick up a fundamental solution that fails to satisfy the NBC. Example 4 replicates the model of McCallum (2004) where the solution obtained by applying the MSV criterion differs from the forward solution, and, consequently, does not satisfy the NBC. The final example reproduces the model of Evans and Honkapohja (2001), where two fundamental solutions pass the E-stability criterion, but only one of them is the forward solution, while the other one does not satisfy the NBC.

Consider the standard New-Keynesian model consisting of aggregate supply (AS), aggregate demand (IS) and monetary policy rule equations proposed by Woodford (2003). The three equations are given by:

$$\pi_t = \delta_1 E_t \pi_{t+1} + \delta_2 \pi_{t-1} + \kappa y_t + v_t \tag{43}$$

$$y_t = \mu_1 E_t y_{t+1} + \mu_2 y_{t-1} - (i_t - E_t \pi_{t+1}) + u_t \tag{44}$$

$$i_t = (1+\beta) E_t \pi_{t+1} + \lambda y_t \tag{45}$$

where π_t is inflation, y_t is the output gap, i_t is the nominal short-term interest rate, and v_t and u_t are white noise supply and preference shocks, respectively. For simplicity, we let the coefficient of the interest rate elasticity be one. Thus the model can be reduced to the following two-variable, two-equation model, by substituting the policy rule into the IS equation:

$$\pi_t = \delta_1 E_t \pi_{t+1} + \delta_2 \pi_{t-1} + \kappa y_t + v_t \tag{46}$$

$$y_t = \mu_1' E_t y_{t+1} + \mu_2' y_{t-1} - \beta' E_t \pi_{t+1} + u_t', \tag{47}$$

where $\mu_1' = \frac{\mu_1}{1+\lambda}$, $\mu_2' = \frac{\mu_2}{1+\lambda}$, $\beta' = \frac{\beta}{1+\lambda}$ and $u_t' = \frac{1}{1+\lambda} u_t$. In matrix form,

$$x_t = A E_t x_{t+1} + B x_{t-1} + C z_t, \tag{48}$$

where $x_t = (\pi_t \; y_t)'$ and $z_t = (v_t \; u_t')'$, and A, B and C are defined as:

$$A = \begin{bmatrix} \delta_1 - \kappa\beta' & \kappa\mu_1' \\ -\beta' & \mu_1' \end{bmatrix}, B = \begin{bmatrix} \delta_2 & \kappa\mu_2' \\ 0 & \mu_2' \end{bmatrix}, C = \begin{bmatrix} 1 & \kappa \\ 0 & 1 \end{bmatrix}. \tag{49}$$

If a real-valued stationary fundamental solution exists, it must be of the following form:

$$x_t = \Omega x_{t-1} + \Gamma \epsilon_t, \tag{50}$$

where (Ω, Γ) must be an element of the following set:

$$\mathcal{A} = \{(\Omega, \Gamma) |\; \Omega = (I_2 - A\Omega)^{-1} B, \Gamma = (I_2 - A\Omega)^{-1}, (\Omega, \Gamma) \in \mathcal{R}^{2\times 2} \times \mathcal{R}^{2\times 2}\}. \tag{51}$$

In this case we have 4 generalized eigenvalues. Typically, in the presence of predetermined variables, the generalized eigenvalues cannot be expressed in closed-form. In order to analyze an example of a closed-form solution, suppose that $\kappa = 0$. Then the inflation process is autonomous so that $w_{12} = 0$ and $\gamma_{12} = 0$, where ω_{ij} is the ij-th element of Ω and γ_{ij} is the ij-th element of Γ for $i, j = 1, 2$. This simplifying assumption is taken to clearly illustrate the implications of the forward solution and other fundamental solutions. The elements of Ω can now be solved analytically and are given by:

$$w_{11} = \frac{1 \pm \sqrt{1 - 4\delta_1\delta_2}}{2\delta_1}, \quad w_{22} = \frac{1 \pm \sqrt{1 - 4\mu_1'\mu_2'}}{2\mu_1'}.$$

Once we select w_{11} and w_{22}, then w_{21} and all the elements in Γ are determined.[16] Furthermore, the generalized eigenvalues are simply two possible values of w_{11} and w_{22}. The 4 eigenvalues are expressed as:

$$\begin{aligned} g_1 &= \frac{1 - \sqrt{1 - 4\delta_1\delta_2}}{2\delta_1}, \quad g_2 = \frac{1 + \sqrt{1 - 4\delta_1\delta_2}}{2\delta_1} \\ g_3 &= \frac{1 - \sqrt{1 - 4\mu_1'\mu_2'}}{2\mu_1'}, \quad g_4 = \frac{1 + \sqrt{1 - 4\mu_1'\mu_2'}}{2\mu_1'} \end{aligned}$$

In this case, we need to choose one eigenvalue between g_1 and g_2, and the other eigenvalue between g_3 and g_4 in order to construct the fundamental solution, because the first two eigenvalues are associated with the first equation and the last two with the second equation. However, it should be noted that since the generalized eigenvalues are associated with all the equations in general, one necessarily chooses two out of four generalized eigenvalues to construct a fundamental solution.

To proceed, let $\Omega(i, j)$ be the fundamental solution associated with g_i and g_j where $i, j = 1, 2, 3, 4$ and $i \neq j$. Note that the fundamental solution obtained by the MSV

[16] Specifically, w_{21} and the elements of Γ are given by $w_{21} = -\frac{\beta' w_{11}^2}{1-\mu_1'(w_{11}+w_{22})}$, $\gamma_{11} = \frac{1}{1-\delta_1 w_{11}}$, $\gamma_{21} = \frac{\mu_1' w_{21} - \beta' w_{11}}{1-\mu_1' w_{22}}\gamma_{11}$ and $\gamma_{22} = \frac{1}{1-\mu_1' w_{22}}$.

criterion is $\Omega(1,3)$ because, according to the MSV criterion, g_1 and g_3 are the generalized eigenvalues that converge to zero when the coefficients of the lagged variables, δ_2 and μ_2', go to zero.

Case 1: This example illustrates a case where there exists a unique stationary fundamental solution which coincides with the forward solution. Suppose that $\delta_1 = 0.58$, $\delta_2 = 1 - \delta_1$, $\mu_1 = 0.604$, $\mu_2 = 1 - \mu_1$, $\beta = 0.1$ and $\lambda = 0.1$. Then μ_1' =0.5491, μ_2' =0.36 and β' =0.0909. The generalized eigenvalues are given by $[g_1\ g_2\ g_3\ g_4] =$ [0.7241 1 0.4940 1.3272]. Note that $g_1 < g_2 = 1$ and $g_3 < 1 < g_4$. Therefore, there are two stable eigenvalues, g_1 and g_3, which are associated with the first and second equations, respectively. Therefore, the set $\mathcal{A}$ has a unique stationary element and Ω is given by:

$$\Omega(1,3) = \begin{bmatrix} 7241 & 0 \\ -0.1440 & 0.4940 \end{bmatrix}.$$

We now apply the forward method. For $k = 30, 50$ and 70, we report the elements of Ω_k:

k	ω_{11}	ω_{21}	ω_{12}	ω_{22}
30	0.7241	−0.1439	0	0.4940
50	0.7241	−0.1440	0	0.4940
70	0.7241	−0.1440	0	0.4940

Therefore $\lim_{k\to\infty} \Omega_k = \Omega^* = \Omega(1,3)$. In this case, convergence of Ω_k implies convergence of Γ_k from equation (37), and therefore the FCC holds. Consequently, $x_t = \Omega^* x_{t-1} + \Gamma^* \epsilon_t$ is the forward solution that satisfies the NBC from Proposition 2.[17] Finally, the forward solution is stationary because the eigenvalues of Ω^*, g_1 and g_3 are smaller than unity

[17]The NBC can be directly verified: Let $Q_k^* \equiv M_k \Omega^{*(k)}$. Then the NBC holds if Q_k^* converges to $0_{2\times 2}$:

in absolute value. Therefore, the unique stationary forward solution coincides with the unique stationary fundamental solution.

Case 2. This example shows a case where multiple stationary fundamental solutions exist and one of them is the forward solution. Suppose that $\lambda = -0.02$ and the remaining parameter values are the same as those in case 1.[18] The generalized eigenvalues are given by $[g_1\ g_2\ g_3\ g_4] = [0.7241\ 1\ 0.7611\ 0.8614]$. Since there are three stable roots, there may be three fundamental solutions associated with (g_1, g_3), (g_1, g_4) and (g_3, g_4). However, the solution associated with (g_3, g_4) cannot be member of $\mathcal{A}$ because it makes the first equation completely disappear from the model.[19]

Now let us apply the forward method to the model. The FCC holds and, consequently, the forward solution exists and is given by $\Omega^* = \begin{bmatrix} 0.7241 & 0 \\ -0.6326 & 0.7611 \end{bmatrix}$. Again $(\Omega^*, \Gamma^*) \in \mathcal{A}$ and it coincides with the fundamental solution $\Omega(1,3)$. This solution is the one associated with the two smallest generalized eigenvalues.

From Proposition 2 and Corollary 2, the other fundamental solution, $\Omega(1,4)$ must violate the NBC. Indeed, if one computes the expectational term $M_k E_t x_{t+k}$ with $\Omega(1,4)$ then, $M_k E_t x_{t+k} = M_k \Omega(1,4)^k x_t$ and $M_k \Omega(1,4)^k$ converges to $\begin{bmatrix} 0 & 0 \\ -1.9887 & 0.1163 \end{bmatrix}$, violating the NBC.

Case 3. This example illustrates a situation where there are multiple stationary

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$1.0e-004 \times$

k	q^*_{11}	q^*_{21}	q^*_{12}	q^*_{22}
30	0.1719	−0.2256	0	0.0000
50	0.0003	−0.0004	0	0.0000
70	0.0000	−0.0000	0	0.0000

where q^*_{ij} is the ij-th element of Q^*_k for $i, j = 1, 2$.

[18] A negative interest rate reaction to the output gap is in general not consistent with a monetary policy aiming at stabilizing the output gap. However, a negative value of λ can be admissible for the stability of the underlying model (see, for instance, Rotemberg and Woodford (1999)).

[19] Technically speaking, it violates the rank condition in McCallum's (1983) sense and in fact, $\Omega(3,4)$ does not exist. We thank Bennett McCallum for pointing us out this fact.

fundamental solutions but none of them is the forward solution. Suppose that $\lambda = -0.02$, $\delta_1 = 0.52$ and that the remaining parameter values are the same as those in case 1. The eigenvalues are given by $[g_1\ g_2\ g_3\ g_4] = [0.9231\ 1\ 0.7611\ 0.8614]$. Note that $g_3 < g_4 < g_1 < g_2 = 1$, and the two smallest eigenvalues, g_3 and g_4, are associated with the second equation. Again, $\Omega(3,4)$ does not exist because the rank condition is violated for this particular solution and is not an element of $\mathcal{A}$. The two solution candidates for Ω are given by:

$$\Omega(1,3) = \begin{bmatrix} 0.9231 & 0 \\ 2.2860 & 0.7611 \end{bmatrix}, \ \Omega(1,4) = \begin{bmatrix} 0.9231 & 0 \\ 0.8712 & 0.8614 \end{bmatrix}.$$

Note that $\Omega(1,3)$ is the stationary fundamental solution obtained by the MSV criterion.

Now consider the forward solution. The FCC does not hold so that the solution obtained above must also violate the NBC. Specifically, Ω_k is given by:

k	ω_{11}	ω_{21}	ω_{12}	ω_{22}
30	0.9161	-6.2866	0	0.7589
70	0.9228	-122.4218	0	0.7611
100	0.9231	-988.5229	0	0.7611

for $k = 30, 70, 100$. While w_{11} and w_{22} converge to g_1 and g_3, w_{21} explodes. Corollary 2 implies that in this case the expectational term, $M_k E_t x_{t+k}$, cannot converge when computed with any of the fundamental solutions. Indeed, the $(2,1)$-th element of $M_k \Omega^{(k)}$

explodes in both cases, although the remaining elements converge to zeros:

k	(2,1)-th element of $M_k\Omega(1,3)^{(k)}$	(2,1)-th element of $M_k\Omega(1,4)^{(k)}$
30	7.6421	9.2798
70	133.4576	135.1001
100	1071.7338	1073.3764

Therefore, the NBC is violated for any fundamental solution including the one obtained via the MSV criterion ($\Omega(1,3)$). This example shows that one should examine whether a solution obtained by other methods satisfies the FCC and, correspondingly, the NBC.

Case 4. This example is taken from McCallum (2004), and shows that even when there is a unique stationary fundamental solution, the solution selected by the MSV criterion can be different, whereas the forward method correctly identifies the unique stationary solution as a forward solution. His second example is of the form (48) with $A = \begin{bmatrix} -0.4 & 0.01 \\ 0.02 & -1.5 \end{bmatrix}$ and $B = \begin{bmatrix} 1.5 & 0.02 \\ 0.01 & 0.2 \end{bmatrix}$. The generalized eigenvalues are $[-3.5563\ 1.0551\ -0.8275\ 0.1610]$. He shows that while the unique determinate solution is the one associated with the two smallest generalized eigenvalues, the MSV criterion selects the solution associated with 1.0551 and 0.1610. We apply the forward method to his example and confirm that the model converges to the solution associated with -0.8275 and 0.1610, the unique forward solution in his example. Indeed, the expectational term dies out as k increases, satisfying the NBC.

Case 5. The final example illustrates a case where there are multiple stationary fundamental solutions, the forward solution exists and several solutions pass the E-stability condition.[20] Evans and Honkapohja (2001) consider a Dornbusch-type model consisting

[20]We verified that the E-stability criterion is consistent in cases 1 through 3. Specifically, the E-stability criterion correctly identifies $\Omega(1,3)$ in cases 1 and 2. In Case 3, where the forward solution does not exist, no solution is E-stable.

of a Phillips curve, an open-economy IS curve, an LM curve and an open-economy parity condition. The model is reproduced as follows:

$$p_t = p_{t-1} + \pi d_t \tag{52}$$

$$d_t = -\gamma(r_t - E_t p_{t+1} + p_t) + \eta(e_t - p_t) \tag{53}$$

$$r_t = \lambda^{-1}(p_t - \vartheta p_{t-1}) \tag{54}$$

$$e_t = E_t e_{t+1} - r_t \tag{55}$$

where p_t is the (log) price level, d_t is (log) aggregate demand, r_t is the nominal interest rate and e_t is the (log) nominal exchange rate.[21]. The model can be reduced to a univariate representation in terms of p_t as:

$$p_t = \beta_1 E_t p_{t+1} + \beta_2 E_t p_{t+2} + \delta p_{t-1}, \tag{56}$$

where $\beta_0 = -(2 + \pi(\gamma + \eta + \gamma/\lambda + \eta/\lambda + \gamma\vartheta/\lambda))$, $\beta_1 = (1 + \pi(2\gamma + \eta + \gamma/\lambda))/\beta_0$, $\beta_2 = -\pi\gamma/\beta_0$, $\delta = (1 + \pi\vartheta(\gamma + \eta)/\lambda)/\beta_0$. They use the parameter values $\pi = 1.5$, $\gamma = 1.5$, $\lambda = 10$, $\vartheta = 1.1$ and $\eta = -0.1$. The fundamental solution (or the perceived law of motion in their terminology) is of the form:

$$p_t = \omega p_{t-1}. \tag{57}$$

There are three stationary solutions for ω: 0.7160, 07721 and 0.9897. One can show that the solutions $\omega = 0.7160$ and $\omega = 0.9897$ are E-stable.[22]

[21]Here we use d_t in the Phillips curve to simplify the analysis while Evans and Honkapohja (2001) use $E_{t-1}d_t$. When lagged expectations are used, the model can be reformulated following Binder and Pesaran (1997) so that the model belongs to the class of (25). We verified that in this instance the same results are obtained.

[22]To see this, note that the mapping from the perceived law of motion to the actual law of motion is given by $T(\omega) = (1 - \beta_1\omega - \beta_2\omega^2)^{-1}\delta$. If the derivative of $T(\omega)$ with respect to ω, $DT(\omega)$ has

Now let us apply the forward method. We rewrite the model (56) in order to make it belong to the class of (25):

$$x_t = AE_t x_{t+1} + Bx_{t-1} \tag{58}$$

where $f_t = E_t p_{t+1}$, $x_t = (p_t \ f_t)'$, $A = \begin{bmatrix} \beta_1 & \beta_2 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix}$. The forward solution exists and it is given by:

$$x_t = \Omega^* x_{t-1} = \begin{bmatrix} 0.7160 & 0 \\ 0.7160^2 & 0 \end{bmatrix} x_{t-1} \tag{59}$$

That is, $p_t = 0.7160 p_{t-1}$ is the forward solution whereas the other E-stable solution $p_t = 0.9897 p_{t-1}$ violates the NBC.

6 Conclusion

This paper generalizes the forward solution method of recursive substitution to RE models with predetermined variables. The essence of our method lies in solving the model forward directly and checking whether the structural model converges to a reduced-form solution. Our method pins down a unique solution by construction and can be easily applied in estimation. It also presents other important advantages with respect to other methods: It can correctly detect the non-existence of a model solution in cases where standard methods state the existence of solutions, and it pins down the correct solution in cases where other methods may fail to do so.

a real part lower than 1, it is said that the solution ω is E-stable. A straightforward computation of $DT(\omega) = (1-\beta_1\omega-\beta_2\omega^2)^{-2}\delta(\beta_1+2\beta_2\omega)$ yields three values for ω: 0.9799, 1.0172 and 0.8923. Therefore, the solutions associated with $\omega = 0.7160$ and $\omega = 0.9897$ are E-stable.

Appendix

A Proof of Proposition 1

1. The forward solution is given by:

$$x_t = \omega^* x_{t-1} + \gamma^* z_t \tag{60}$$

Since $(\omega^*, \gamma^*) \in \mathcal{A}$, the forward solution is a fundamental solution to the model and therefore, it must solve the forward representation of the model (15) for all $k = 1, 2, ...$ Therefore, it must be true that $\lim_{k\to\infty} m_k E_t x_{t+k} = 0$. We can also directly prove this fact. From the forward representation,

$$m_k E_t x_{t+k} = m_k (\omega^*)^k x_t + m_k \sum_{i=1}^{k} (\omega^*)^{k-i} (\gamma^*) \rho^i z_t \tag{61}$$

Plugging $m_k E_t x_{t+k}$ into (15) and matching the coefficients of x_{t-1} and z_t yields:

$$(1 - m_k(\omega^*)^k)\omega^* = \omega_k \tag{62}$$

$$(1 - m_k(\omega^*)^k)\gamma^* = \gamma_k + m_k \sum_{i=1}^{k} (\omega^*)^{k-i} (\gamma^*) \rho^i. \tag{63}$$

Since $\lim_{k\to\infty} \omega_k = \omega^*$, $\lim_{k\to\infty} m_k(\omega^*)^k = 0$ from equation (62). But then, the LHS of equation (63) converges to γ^* and since and $\lim_{k\to\infty} \gamma_k = \gamma^*$ on the RHS, $m_k \sum_{i=1}^{k} (\omega^*)^{k-i} (\gamma^*) \rho^i$ must converge to 0. This implies that $m_k E_t x_{t+k}$ must converge to 0 for any given x_{t-1} and z_t, implying that the forward solution satisfies the NBC. Since the pair (ω_k, γ_k) is unique and real-valued given the structural parameters a and b, the limiting values (ω^*, γ^*) are also unique and real-valued.

2. When the FCC does not hold, the pair (ω_k, γ_k) is either not well-defined if the regularity condition is violated or does not converge even if the regularity condition is met. Consequently, there is no forward solution and for any other solution, fundamental or non-fundamental, $\lim_{k\to\infty} m_k E_t x_{t+k}$ is not well-defined or it does not converge, implying the violation of the NBC. When the FCC holds, suppose that the NBC holds for a solution, fundamental or non-fundamental, different from the forward solution. Since the solution must solve (20), (20) becomes the forward solution under the NBC, i.e., $\lim_{k\to\infty} m_k E_t x_{t+k} = 0$, which is a contradiction to the fact that the solution is different from the forward solution. *Q.E.D.*

B Proof of Corollary 1

1. Since the FCC holds, the forward solution exists and $(\omega^*, \gamma^*) \in \mathcal{A}$. Therefore, it must be a member of $\mathcal{A}$, thus $\mathcal{A}$ is not empty. In this model, $\mathcal{A}$ has at most two elements. If the other solution exists, then there are two elements of $\mathcal{A}$, $(\omega(s), \gamma(s))$, for $s = 1, 2$. Each solution must solve (15) for all $k = 1, 2, ...$ Since $m_k E_t x_{t+k} = m_k(\omega(s))^k x_t + m_k \sum_{i=1}^{k} \omega(s)^{k-i}\gamma(s)\rho^i z_t$ for each $(\omega(s), \gamma(s)) \in \mathcal{A}$, substituting this into equation (15) and matching the coefficients of x_{t-1} and z_t yields:

$$(1 - m_k\omega(s)^k)\omega(s) = \omega_k \quad (64)$$

$$(1 - m_k\omega(s)^k)\gamma(s) = \gamma_k + m_k \sum_{i=1}^{k} \omega(s)^{k-i}\gamma(s)\rho^i \quad (65)$$

for both s and all k. This implies that $m_k E_t x_{t+k}$ is solution-dependent and that $m_k\omega(s)^k$ must converge to a constant for each s, as the right-hand-side of (64) converges to ω^*. Note that ω^* must be either $\omega(1)$ or $\omega(2)$ because $(\omega^*, \gamma^*) \in A$, implying that only one of the $m_k\omega(s)^k$ converges to zero. Since $|\omega(1)| < |\omega(2)|$, it must be true that $0 =$

$|\lim_{k\to\infty} m_k\omega(1)^k| < |\lim_{k\to\infty} m_k\omega(2)^k| < \infty$. This proves that $\omega^* = \omega(1)$. Then, from equation (65), $\lim_{k\to\infty} m_k \sum_{i=1}^k \omega(1)^{k-i}\gamma(1)\rho^i = 0$. Hence, $\gamma^* = \gamma(1)$. Therefore, the forward solution corresponds to the smallest root of ω. Since $m_k\omega(2)^k$ must also converge, it is not to zero because otherwise, it is a contradiction to $\omega(2) = \omega^*$. Let $l_x(2) = \lim_{k\to\infty} m_k\omega(2)^k$. Then from equation (65), $(1-l_x(2))\gamma(s) = \gamma^* + l_z(2)$ where $l_z(2) = \lim_{k\to\infty} m_k \sum_{i=1}^k \omega(s)^{k-i}\gamma(s)\rho^i$. Therefore, $\lim_{k\to\infty} m_k E_t x_{t+k} = l_x(2)x_t + l_z(2)z_t \neq 0$.

2. This is an immediate consequence of Proposition 1. Suppose that the FCC does not hold. Then either ω_k or γ_k or both do not converge (explodes or oscillates). If $\mathcal{A}$ is not-empty, then $(\omega(s), \gamma(s))$ are constants and consequently, either $m_k\omega(s)^k$ or $m_k \sum_{i=1}^k \omega(s)^{k-i}\gamma(s)\rho^i$ or both explode or oscillate from (64) and (65), implying that $m_k E_t x_{t+k}$ explodes or oscillates when when expectations are formed with any $(\omega(s), \gamma(s))$ for $s = 1, 2$. *Q.E.D.*

C Proof of the Claim

The model is given by:

$$\begin{aligned} x_t &= \alpha + AE_t x_{t+1} + Bx_{t-1} + Cz_t & (66) \\ &= M_1 E_t x_{t+1} + c_1 + \Omega_1 x_{t-1} + \Gamma_1 z_t. & (67) \end{aligned}$$

Suppose that there exist sequences of matrices, $\{c_{k-1}, M_{k-1}, \Omega_{k-1}, \Gamma_{k-1}\}$ for some $k > 1$ such that:

$$x_t = M_{k-1}E_t x_{t+k-1} + c_{k-1} + \Omega_{k-1}x_{t-1} + \Gamma_{k-1}z_t. \quad (68)$$

Shifting this equation forward one period and taking conditional expectations,

$$E_t x_{t+1} = M_{k-1}E_t x_{t+k} + c_{k-1} + \Omega_{k-1}x_t + \Gamma_{k-1}F z_t \quad (69)$$

by the law of iterative expectations. Substituting (69) into (66), we have:

$$(I_n - A\Omega_{k-1})x_t = AM_{k-1}E_t x_{t+k} + (\alpha + Ac_{k-1}) + Bx_{t-1} + (C + A\Gamma_{k-1}F)z_t. \tag{70}$$

Provided that $(I_n - A\Omega_{k-1})$ is non-singular, there exists a set $\{M_k, c_k, \Omega_k, \Gamma_k\}$, where these matrices are given by (34) through (37). Therefore, if $(I_n - A\Omega_{k-1})$ is invertible for all k, the sequences of $\{M_k, c_k, \Omega_k, \Gamma_k\}$ are well defined. *Q.E.D.*

D Proof of the Proposition 2

The proof of the Proposition 2 is essentially the same as that of Proposition 1. Simply replace the lower case letters with the corresponding upper case letters except for F instead of ρ. The results are independent of the existence of constants, α. *Q.E.D.*

E Proof of the Corollary 2

1. Since the FCC holds, the forward solution exists and $(c^*, \Omega^*, \Gamma^*) \in \mathcal{A}$. Therefore, it must be a member of $\mathcal{A}$, say, $(c(1), \Omega(1), \Gamma(1))$. Thus $\mathcal{A}$ is not empty. Suppose there are other solutions in $\mathcal{A}$, $(c(s), \Omega(s), \Gamma(s))$, for $s = 2, 3, .., S$. Then each solution must solve (33) for all $k = 1, 2, ..$ and for all $s = 1, 2, ..., S$. Since $M_k E_t x_{t+k} = M_k \sum_{i=1}^{k} \Omega(s)^{i-1} c(s) + M_k(\Omega(s))^k x_t + M_k \sum_{i=1}^{k} \Omega(s)^{k-i}\Gamma(s)F^i z_t$, substituting this into

equation (33) and matching the constants, the coefficients of x_{t-1} and z_t yields:

$$(1 - M_k(\Omega(s))^k)c(s) = c_k + M_k \sum_{i=1}^{k} \Omega(s)^{i-1} c(s) \tag{71}$$

$$(1 - M_k(\Omega(s))^k)\Omega(s) = \Omega_k \tag{72}$$

$$(1 - M_k(\Omega(s))^k)\Gamma(s) = \Gamma_k + M_k \sum_{i=1}^{k} \Omega(s)^{k-i} \Gamma(s) F^i \tag{73}$$

for both s and all k. In the limit, $\lim_{k\to\infty} c_k = c(1)$, $\lim_{k\to\infty} \Omega_k = \Omega(1)$ and $\lim_{k\to\infty} \Gamma_k = \Gamma_\epsilon(1)$. This implies that from equation (72), $L_x(s,k)$ converges for all s. Therefore, the left-hand-side of (71) and (73) must converge, implying that $M_k \sum_{i=1}^{k} \Omega(s)^{i-1} c(s)$ and $M_k \sum_{i=1}^{k} \Omega(s)^{k-i} \Gamma(s) F^i$ also converges. For all $s \neq 1$, $M_k(\Omega(s))^k$ in equation (72), cannot converge to zeros because $\Omega(s) \neq \Omega^* = \Omega(1)$. $M_k \sum_{i=1}^{k} \Omega(s)^{i-1} c(s)$ (or $M_k \sum_{i=1}^{k} \Omega(s)^{k-i} \Gamma(s) F^i$) converges to zeros if $\alpha = 0$ (or $F = 0$), but the whole expectational term, $M_k E_t x_{t+k}$ in (33) cannot converge to zeros, because otherwise, it is a contradiction to $\Omega(s) \neq \Omega^*$. Therefore, $\lim_{k\to\infty} M_k E_t x_{t+k} \neq 0_{n\times 1}$ for all $s \neq 1$.

2. This is an immediate consequence of Proposition 1. Suppose that the FCC does not hold. Then either Ω_k or Γ_k or both do not converge. If $\mathcal{A}$ is not-empty, then $(\Omega(s), \Gamma(s))$ are constants for all $s = 1, 2, .., S$, and consequently, either $M_k(\Omega(s))^k$ or $M_k \sum_{i=1}^{k} \Omega(s)^{i-1} c(s)$ or $M_k \sum_{i=1}^{k} \Omega(s)^{k-i} \Gamma(s) F^i$ explodes as well, from (72), (71) and (73), implying that $M_k E_t x_{t+k}$ explodes for any $(\omega(s), \gamma(s))$ in $\mathcal{A}$. $Q.E.D.$

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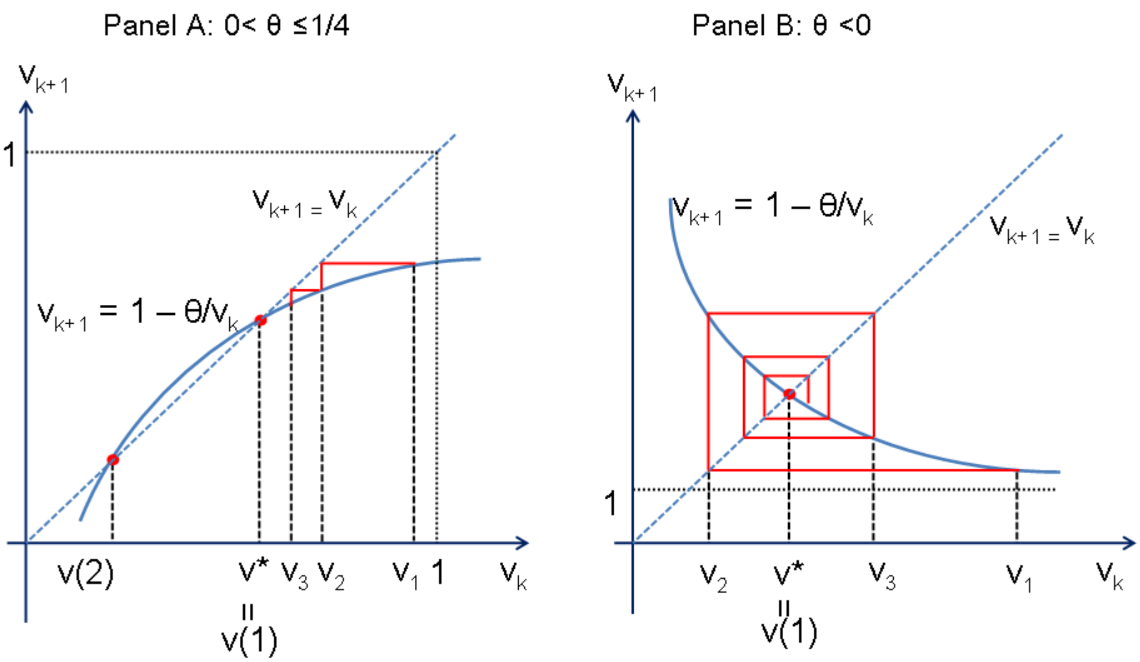
Figure 1: The Existence of the Forward Solution

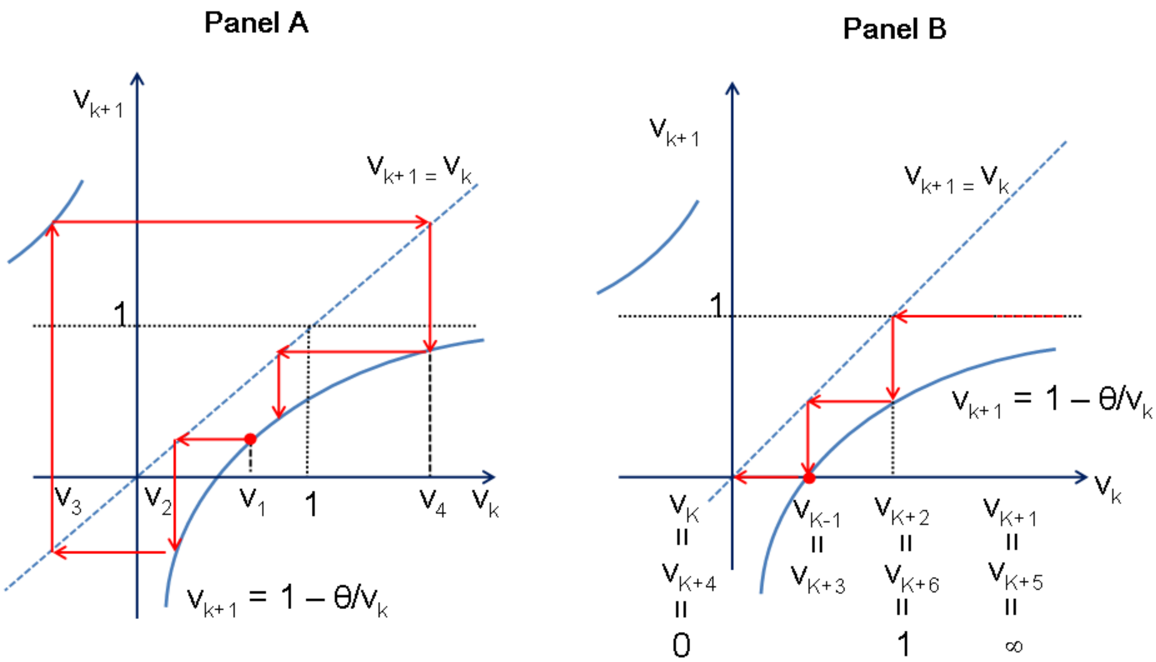


This figure shows graphically the recursive path of the regularity condition for the existence of the forward solution in this univariate model:

$$x_t = aE_t x_{t+1} + bx_{t-1} + \epsilon_t$$

Panel A describes the convergence path associated with $0 < \theta \leq \frac{1}{4}$, whereas Panel B describes the path for $\theta < 0$, where $\theta = ab$. The regularity condition is that $\upsilon_k \neq 0 \quad \forall k$, and $\upsilon_1 = 1 - \theta$.

Figure 2: The Non-Existence of the Forward Solution



This figure shows graphically the recursive path of the regularity condition for the non-existence of the forward solution in this univariate model:

$$x_t = aE_t x_{t+1} + bx_{t-1} + \epsilon_t$$

In both panels $\theta > \frac{1}{4}$ where $\theta = ab$. The regularity condition is that $\upsilon_k \neq 0 \quad \forall k$, and $\upsilon_1 = 1 - \theta$. Panel A illustrates the case where the regularity condition holds, but υ_k oscillates. Panel B illustrates the case where the regularity condition is violated.