Determination of risk pricing measures from market prices of risk

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Abstract

A new insurance provider or a regulatory agency may be interested in determining a risk measure consistent with observed market prices of a collection of risks. Using a relationship between distorted coherent risk measures and spectral risk measures, we provide a method for reconstructing distortion functions from the observed prices of risk. The technique is based on an appropriate application of the method of maximum entropy in the mean, which builds upon the classical method of maximum entropy.

Key words. Distortion function, Spectral measures, Risk Aversion Function, Maximum entropy in the mean, Inverse problems.

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1 Introduction

The problem of pricing actuarial risk has received a great deal of attention in recent years, generating a large amount of theoretical and practical work. A good account of the traditional and modern pricing methods appears in the book by Kaas, Goovaerts, Dhaene and Denuit (Kaas et al. (2005)). Among pricing measures the class of *distorted risk measures* is popular and appealing. For further details and the general philosophy of distorted risk measures the interested reader may consult Wang (1996), Wang, Young and Panjer (1997), and Wang and Young (1998), where, building on Quiggin (1982), Yaari (1987), and Schmeidler (1989), an axiomatic characterization and many applications to insurance can be found. Recent work on the use of actuarial risk measures for financial derivatives pricing is due to Goovaerts and Laeven (2008) (see also the companion paper Goovaerts, Kaas, Laeven and Tang (2004)), and for the relationship between risk measurement and decision making we refer to Goovaerts, Kaas and Laeven (2008). Denneberg (1997) introduced the distorted probability measure by means of the Choquet integral. The notion of distorted risk measure is a specific example of that concept. A distorted risk measure can be defined as the expected value of a random financial outcome where the expectation is taken under a transformation of the cumulative density function. Distortion risk measures are extremely flexible and simple to use to price risks.

This paper addresses a different issue. Imagine that a new participant in the insurance services business wants to know how his competitors price risk. Or imagine that you already know the prices of some risks and that you want to devise a way to price other risks that is consistent with the prices of the already priced risks. It turns out that the methodology of risk distortion functions is also of assistance with these problems, or actually, the relationship between pricing risk with concave distortion functions and the theory of coherent (spectral) risk measures is of great use.

For the measurement of market risk, coherent risk measures provide a class of measures satisfying a conceptual desideratum that a risk measure may satisfy. Artzner, Delbaen, Eber and Heath (1999) proposed several properties that a risk measures must satisfy, thus establishing the notion of coherent risk measure.

However, there are also other risk measures available, e.g., deviation, convex and spectral risk measures. See Acerbi 2002, Fölmer and Schied (2002), Rockafellar et.al (2006) and Dhaene, Laeven, Vanduffel, Darkiewicz and Goovaerts (2008). These families of risk measures are interconnected and some among these have a direct relationship to coherent measures, in particular spectral risk measures, which are interesting to us because they provide a bridge between coherent risk measures and distorted risk measures. Some interesting coherent risk measures, like Conditional Value at Risk, are distorted risk measures, as we shall see in Section 2. Another popular distortion function is the Wang Transform, which when used to price financial derivatives, reproduces well-known results such as CAPM and Black-Scholes as special cases, see Wang (2000). Other distortion functions used to value insurance premiums are the dual-power distortion and the proportional hazards (PH) distortion (Wang, 1996), which are special cases of the beta distortion function. The proportional hazard distortion functions are a special subclass of coherent distortion functions that relate nicely to spectral risk measures.

Clearly, the choice of a distortion function defines a pricing procedure. But, there are no rules to decide on how one must define the distortion function. We only know that it amounts to a re-weighting of the initial distribution of the liabilities. Sometimes, the choice of the distortion function depends on the generic properties that we want the risk measure to satisfy.

In this paper we provide a nonparametric method for the construction of distortion functions from the observed prices of risk. But to apply our method, we must assume that we have enough data to infer the distribution function of the liabilities to be priced. With the method we propose, the distortion function is not chosen by an ad-hoc procedure, but to match the market prices of risk. Our method consists of an application of the method of maximum entropy in the mean to obtain the distortion function.

From the mathematical point of view, our technique falls within the category of solving Fredholm equations (see Dieudonné (1960)), and is classified as a non parametric technique by statisticians, but is different form the usual way maxentropic techniques that are used in for solving generalized moment problems, and in particular is different from the way the maximum entropy method has been used in finance for reconstructing risk neutral densities. The method of maximum entropy in the mean builds upon the standard method of maximum entropy, but is a completely different in sprit from the standard method of maximum entropy in the way it handles the constraints imposed to fit the model to the prices of benchmark instruments.

As option prices provide a source of information to estimate risk-neutral densities of the underlying asset price, market prices of risk provide information to obtain risk distortion functions of risk, while the statistical nature of the liability is assumed as known. Many methods to estimate risk neutral distributions exist, for example, parametric density specifications including a mixture of log-normals (Ritchey, 1990), a generalized beta (Anagnou-Basioudis et al., 2005). Other approaches are multi-parameter discrete distributions (Jackwerth and Rubinstein, 1996) and densities from smile functions defined by splines (Bliss and Panigirtzoglou, 2002). For a description of a nonparametric procedure, consider Aït-Sahalia and Lo (1998). As a very short list of references of the application of the method on maximum entropy to obtain risk neutral measures consider Breeden and Litzenberger (1978), Gerber and Shiu (1994) in which finance and actuarial sciences are related, or Stutzer (1996), Frittelli (2000) or more recently Choulli and Sticker (2005) to name but a few.

This paper is organized as follows: In Section 2 we introduce the concept of a distortion measure and recall the relationship of these measures to coherent and spectral risk measure. In Subsection 2.1 and using the relationship of spectral and distortion risk measures we establish the Fredholm equation which relates the distortion function with the observed prices of risk. In Section 3 we present the method of maximum entropy in the mean (MEM), which consists of a technique for transforming an ill-posed linear problem with convex constraints into a simpler (possibly unconstrained) but non-linear minimization problem. In Section 4 we present numerical examples. Finally, Section 5 concludes the paper.

2 Preliminaries

We consider a one period market model $(\Omega, \mathcal{F}, P)$. The information about the market, that is the σ-algebra $\mathcal{F}$, is generated by a finite collection of random variables, $\mathcal{F} = \sigma(S_0, S_1, ...S_N)$, where the $\{S_j \,|\, j = 0, ..., N\}$ are the basic liabilities traded in the market. We shall model the present worth of our position by $X \in \mathcal{L}_2(P)$ (the square P-integrable functions), that is, all random variables with finite variance. Artzner et al (1999) and Delbaen (2003) suggested a set of properties that a risk measure should satisfy. The risk measures satisfying these properties are called *coherent risk measures.* In these papers the risk measure was assigned to a random variable X describing the worth of a financial position which could be negative or positive. For actuarial applications, where X denotes liabilities it makes sense to consider positive valued random variables, and it makes sense to modify the definition of coherent measure a bit. For that we follow Wirch and Hardy (1999)

Definition 2.1 *A coherent risk measure is defined to be a function ρ defined on the class of positive bounded or the class of positive random variables with finite variance, that satisfies the following axioms:*

1. *A risk measure should be bounded below by the expected value of the loss and above by the maximal loss: $E[X] \leq \rho(X) \leq esssup(X)$.*

2. *Scale and translation Invariance: For any $X \in \mathcal{L}_2$ and $a, \lambda \in \mathbb{R}_+$ we have $\rho(\lambda X + a) = \lambda\rho(X) + a$.*

3. *No unjustified loading, or the risk measure of a certain loss equals the loss. That is, If $X = 1$ a.s., then $\rho(1) = 1$.*

4. *Monotonicity: For any* X *and* $Y \in \mathcal{L}_2$*, such that* $X \leq Y$ *then* $\rho(X) \leq \rho(Y)$*.*

5. *Subadditivity: For any* X *and* $Y \in \mathcal{L}_2$*,* $\rho(X+Y) \leq \rho(X) + \rho(Y)$*.*

One example of coherent risk measures is the Conditional Value at Risk (CVaR). This measure indicate the expected loss incurred in the worst cases of the position. It is the most popular alternative to the Value at Risk, which popular and useful as it is, does not satisfy the coherence axioms because it may fail to be subadditive.

$$CVaR_\alpha(X) = E[X \mid X \geq VaR_\alpha(X)] = \frac{1}{1-\alpha}\int_{1-\alpha}^{1} q_X(t)dt\ .$$

where $q_X(t) = VaR_t(X) = \sup\{x : P[X > x] > 1-t\} = \inf\{x : P(X \leq x) \geq t\}$. Lets us now turn our attention to distortion functions, to their associated Choquet integrals and to the risk measures that they define.

Definition 2.2 *We shall say that* $g : [0,1] \to [0,1]$ *is a distortion function if*

1. $g(0) = 0$ *and* $g(1) = 1$*.*

2. g *is non-decreasing function.*

Let X *be random variable describing losses, having decumulative distribution function* $P(X > x) = S_X(x), (0 \leq x < \infty)$*, the Choquet integral with respect to distortion operator* g *is defined by*

$$H_g[X] = \int_0^\infty g[S_X(x)]dx$$

The Choquet integral introduced above is used to define a risk pricing measure by setting, $\rho_g(X) = H_g[X]$. Thus, distorted risk pricing measures can be thought of as the expected value of a random financial outcome where the expectation is taken under a transformation of the cumulative density function. The relationship between coherence and distortion was studied in Hardy and Wirch (2001) and later generalized Reesor and McLeish (2003).
Some other commonly employed distortion functions are contained in the following list. We shall use them below to construct examples.

1. Dual-power functions:

$$g(u) = 1 - (1-u)^{\nu} \tag{2.1}$$

   with $\nu \geq 1$.

2. Proportional Hazard transforms:

$$g(u) = u^{\frac{1}{\gamma}} \tag{2.2}$$

   with $\gamma \geq 1$.

3. Wang's distortion function:

$$g_{\alpha}(u) = \Phi[\Phi^{-1}(u) + \alpha], \quad u \in (0,1) \tag{2.3}$$

   where Φ is the standard Normal distribution and $\alpha \in \mathbb{R}$.

We should also mention that $CVaR_{\alpha}$ is a distortion risk measure with respect to the following distortion function:

$$g(x) = \begin{cases} \frac{x}{1-\alpha} & \text{if } x \leq \alpha \\ 1 & \text{if } x \geq 1-\alpha \end{cases}. \tag{2.4}$$

Spectral risk measures were proposed by Acerbi, see for example Acerbi (2002), and they can be expressed as general convex combination of the quantiles function of the risk. For actuarial applications it is convenient to change these conventions a bit.

Definition 2.3 *An element $\phi \in \mathcal{L}_1([0,1])$ (the class of Lebesgue integrable functions) is called an admissible risk spectrum if*

1. $\phi \geq 0$

2. ϕ is increasing

3. $\|\phi\| = \int_0^1 \phi(t)dt = 1$.

Definition 2.4 *Let $\phi \in \mathcal{L}_1([0,1])$ be an admissible risk spectrum. The risk pricing measure*

$$\rho_\phi(X) = \int_0^1 q_X(u)\phi(u)du$$

is called the spectral risk measure generated by ϕ.

The function ϕ is called the Risk Aversion Function and assigns, in fact, different weights to different p-confidence level of the left tail. Any rational investor can express her subjective risk aversion by drawing a different profile for the weight function ϕ. The spectral risk measures are a subset of coherent risk measures as Acerbi proves. Specifically, a spectral measure can be associated with a coherent risk measures that has two additional properties, law invariance and comonotone additivity. The risk aversion functions corresponding to the distortion functions listed above are easy to compute.

1. Dual-power risk aversion functions:

$$\phi(u) = \nu u^{\nu-1}. \tag{2.5}$$

2. Proportional Hazard risk aversion function:

$$\phi(u) = \frac{1}{\gamma}(1-u)^{\frac{1}{\gamma}-1}. \tag{2.6}$$

3. Wang's risk aversion function:

$$\phi_\alpha(u) = e^{-\alpha\Phi^{-1}(u)-\alpha^2/2}. \tag{2.7}$$

It is also of interest to note that Conditional Value at Risk can be thought of as a spectral risk measure defined by the Risk Aversion Function:

$$\phi(p) = g'(1-u) = \frac{1}{1-\alpha}1_{\{\alpha\leq p\leq 1\}} \tag{2.8}$$

which according to Theorem (2.1) below, is obtained from (2.4) as indicated. Both in Fölmer and Schied (2004) and in Gzyl and Mayoral (2007), proofs of the relationship between the admissible spectral function and distortion functions are presented. For this paper we recall the appropriate variation on the theme, to wit

Theorem 2.1 *Let g a concave distortion function, and let H_g be the associated distorted risk pricing measure. Then $\phi(u) = g'(1-u)$ defines a spectral measure ρ_ϕ such that $\rho_\phi(X) = H_g(X)$.*

To indicate why this theorem must be true, consider the case where the risk X is a continuous random variable having a strictly positive density with respect to the Lebesgue measure on $[0, \infty)$. The computation goes as follows. The steps consist of integration by parts and changing variables according to $u = 1 - S_X(x)$.

$$\begin{aligned} H_g(X) &= \int_0^\infty g(S_X(x))dx = \int_0^\infty x f_X(x) g'(S_X(x))dx \\ &= \int_0^1 S_X^{-1}(1-u)g'(1-u)du = \int_0^1 q_X(u)\phi(u)du. \end{aligned}$$

2.1 Problem statement

Given the identity relating the distorted price of a positive risk X having a continuous distribution function $F(x)$

$$\begin{aligned} E^*[X] &= \int_0^\infty x dF_X^*(x) = \int_0^\infty g(1-F(x))dx = H_g(X) \\ &= \int_0^1 q_X(u)g'(1-u)du = \rho_\phi(X) \end{aligned}$$

where $F^*(x) = 1 - g(1 - F(x))$, we state our basic problem as, given the market price π_i of a finite collection of risk positions X_i for $i = 1, ..., M$, find a function spectral risk aversion function ϕ such that

$$\pi_i = \rho_\phi(X_i) = \int_0^1 q_{X_i}(u)\phi(u)du, \quad i = 1...., M \tag{2.9}$$

where to accommodate the condition $\int_0^1 \phi(u)du = 1$ we choose X_M such that $q_{X_M}(u) = 1$ and $\pi_M = 1$.

How to solve Fredholm equations like (2.9) with maximum entropy in the mean was first described in Gamboa and Gzyl (1997). To actually solve this problem in practice, the first step consists of a discretization procedure. For that we consider a partition of $[0, 1]$ at points $u_j = j/N$. The choice of N depends on the known variability of $q_X(u)$ in $[0, 1]$. Let us define the $M \times N$ matrix $\mathbf{B}$ by setting $B_{i,j} = q_{X_i}(u_j)/N$, for $i = 1, ..., M$ and $j = 1, ..., N$. Set $\phi(a_j) = \phi_j$, where

$a_j = \frac{1}{2}(u_j + u_{j-1})$ and $u_0 = 0$. With all this, the problem (2.9) can be restated as: Solve

$$\mathbf{B}\phi = \pi; \quad \phi \in \mathbf{K_o}. \tag{2.10}$$

where the constraint set $\mathbf{K}_0 \subset \mathbb{R}^N$ is a convex set defined in this case by

$$\mathbf{K_o} = \{(\phi_1, ..., \phi_N) \,|\, \phi_1 < ... < \phi_j < \phi_{j+1} < ... < \phi_N\}.$$

To simplify the description of the constraints, we set $\phi_1 = \psi_1$, $\phi_2 = \psi_1 + \psi_2, ...$ and $\phi_N = \psi_N + ... + \psi_1$, or $\phi = \mathbf{C}\psi$ where $\mathbf{C}$ is the obvious lower diagonal matrix describing the change of coordinates. Setting $\mathbf{A} = \mathbf{BC}$ we can restate our discretized problem as

$$\mathbf{A}\psi = \pi; \quad \psi \in \mathbf{K}. \tag{2.11}$$

where now the convex constraint set is $\mathbf{K} = \mathbb{R}^N_{++}$, i.e., the positive orthant in $\mathbb{R}^N$. Clearly, once the vector ψ is at hand, the ϕ is easily recovered.

3 The basics of maximum entropy in the mean

3.1 Basic methodology

The method of maximum entropy in the mean (MEM) is a technique for transforming an ill-posed linear problem with convex constraints like (2.11) into a simpler (possibly unconstrained) but non-linear optimization problem. The number of variables in the auxiliary problem being equal to the number of equations in the original problem, M in our case. To carry out the transformation one thinks of the ψ_j there as the expected value of a random variable Ψ_j with respect to some measure Q which is to be determined. The basic datum is a sample space $(\Omega, \mathcal{F})$ on which Ψ is to be defined. In our setup the natural choice is to take $\Omega = \mathbf{K}$, $\mathcal{F} = \mathcal{B}(\mathbf{K})$, the Borel subsets of $\mathbf{K}$, and $\Psi = \mathbf{id_K}$ as the identity map.

To continue we need to select a reference or prior (but not in the Bayesian sense) measure $dQ^o(\xi)$ on $(\Omega, \mathcal{F})$. The only restriction that we impose on it is that the closure of the convex hull of $supp(Q)$ is $\mathbf{K}$. This prior measure embodies knowledge that we may have about ψ. And to get going we define the class

$$\mathbb{P} = \{Q \,|\, Q << Q^o;\ AE_Q[\Psi] = \pi\}. \tag{3.1}$$

and observe now that the algebraic problem (2.11) is transformed into the problem consisting of finding a measure $Q \in \mathbb{P}$. Note that for any $Q \in \mathbb{P}$ having a strictly positive density $\rho = \frac{dQ}{dQ^o}$, then $E_Q[\Psi] \in \text{int}(\mathbf{K})$. This follows since expectation is basically a linear convex combination. The procedure to explicitly produce such Q's is known as the method of maximum entropy, exponential tilting or the Esscher transform. The first step of which is to assume that $\mathbb{P} \neq \emptyset$, which amounts to say that our problem has a solution and define

$$S_Q^o : \mathbb{Q} \to [-\infty, \infty)$$

by the rule

$$S_Q^o(Q) = -\int_\Omega \ln(\frac{dQ}{dQ^o})dQ \tag{3.2}$$

whenever the function $\ln(\frac{dQ}{dQ^o})$ is Q-integrable and $S_Q^o(Q) = -\infty$ otherwise. This entropy functional is concave on the convex set $\mathbb{P}$. To guess the form of the density of the measure Q^* that maximizes S_Q^o is to consider the class of exponential measures on Ω defined by

$$dQ_\lambda = \frac{e^{-<\lambda,\mathbf{A}\Psi>}}{Z(\lambda)}dQ^o \tag{3.3}$$

where the normalization factor is

$$Z(\lambda) = E_Q^o[e^{-<\lambda,\mathbf{A}\Phi>}].$$

Here $\lambda \in \mathbb{R}^M$. If we define the dual entropy function

$$\Sigma(\lambda) : \mathcal{D}(Q) \to (-\infty, \infty]$$

by the rule

$$\Sigma(\lambda) = \ln Z(\lambda) + < \lambda, \pi > \tag{3.4}$$

or $\Sigma(\lambda) = \infty$ whenever $\lambda \notin \mathcal{D}(Q) \equiv \{\mu \in \mathbb{R}^M \mid Z(\mu) < \infty\}$.

It is easy to prove that, $\Sigma(\lambda) \geq S_Q(P)$ for any $\lambda \in \mathcal{D}(Q)$, and any $P \in \mathbb{P}$. Thus if we were able to find a $\lambda^* \in \mathcal{D}(Q)$ such that $P_{\lambda^*} \in \mathbb{P}$, we would have solved our problem. To find such a λ^* it suffices to minimize (the convex function) $\Sigma(\lambda)$ over (the convex set) $\mathcal{D}(Q)$. We leave for the reader to verify that if the minimum is reached in the interior of $\mathcal{D}(Q)$, then $P_{\lambda^*} \in \mathbb{P}$.

3.2 Two possible solution schemes

As is clear from the statement of (2.11), the actual implementation scheme depends on the assumptions that we place on the constraint set $\mathbf{K}$. Here we shall propose two possible alternatives consisting of assuming $\mathbf{K}$ to be bounded or unbounded. And once this aspect of the modeling process is decided, the other degree of freedom that one has corresponds to the choice of the reference measure Q^o.

3.2.1 The bounded case

This choice is adequate when we have reasons to assume the ϕ_j are bounded, which is a natural assumption. Thus let us assume that for appropriate a and b, we know that $a \leq \psi_j \leq b \, \forall j$. This amounts to assuming that $\mathbf{K} = [a,b]^N$. We should add that all the $a's$ and $b's$ could be assumed different with no problem at all. Also, since any point in $[a,b]$ is a convex combination of the end points, a simple assumption consists of putting

$$dQ^o(\xi) = \prod_{j=1}^{N} (p\delta_a(d\xi_j) + q\delta_b(d\xi_j)). \tag{3.5}$$

We use the standard notation $\delta_a(dx)$ to denote the unit point mass measure concentrated at a (the Dirac measure at a). The parameters p, q are such that $p + q = 1$ and they reflect the possible bias of the $\psi_j's$ towards one of the ends of the interval. When no bias is assumed, one chooses $p = 1/2$. A similar assumption would consist of choosing a uniform distribution on $[a,b]$.

The next step consists of computing the normalization factor $Z(\lambda)$. Clearly

$$Z(\lambda) = \prod_{j=1}^{N} \zeta((\mathbf{A}^*\lambda)_j)),$$

where $\zeta(\tau)$ is the Laplace transform of $p\delta_a(dx) + q\delta_b(dx)$, that is

$$\zeta(\tau) = \int e^{-x\tau} p\delta_a(dx) + q\delta_b(dx) = pe^{-a\tau} + qe^{-b\tau}.$$

The following step has to carried out numerically. It consists of finding the minimizer λ^* in (3.4). Once that is accomplished, it is easy to see that the maxentropic reconstruction ψ^* is given by

$$\psi_j^* = ap_j^* + bq_j^*, \tag{3.6}$$

where

$$p_j = \Big(\frac{e^{-a(A^*\lambda^*)_j}}{e^{-a(A^*\lambda^*)_j} + e^{-b(A^*\lambda^*)_j}}\Big), \quad q_j^* = 1 - p_j^*.$$

And now the ϕ_j must be recovered from the ψ_j as described at the end of Section 2.1. Notice that the MEM procedure has shifted the parameters of the distribution. That is the post-data, maximum entropy distribution Q^* is different from the prior (reference) measure Q^o in two respects. First, the components of ξ are no longer independent (the distribution is no a product on 1-dimensional distributions), and second, the original bias in the choice of p and q has been modified.

3.2.2 The unbounded case

Now we shall see one way of solving (2.11) when the constraint space is $\mathbf{K} = \mathbb{R}^N_{++}$. Now we may consider a product of $\Gamma(a, b)$ as our reference measure, that is

$$dQ^o(\xi) = \prod_{j=1}^{N} \Big(\frac{b^a \xi_j^{a-1} e^{-b\xi} d\xi_j}{\Gamma(a)}\Big). \tag{3.7}$$

As above, the next step consists of finding the normalization function $Z(\lambda)$. Again, our assumption leads to a product

$$Z(\lambda) = \prod_{j=1}^{N} \zeta((\mathbf{A}^*\lambda)_j)),$$

where now the $\zeta(\tau)$ are Laplace transforms of the $\Gamma(a,b)$, that is $\zeta(\tau) = \left(\frac{b}{\tau+b}\right)^a$, and therefore

$$Z(\lambda) = \prod_{j=1}^{N} \Big(\frac{b}{(\mathbf{A}^*\lambda)_j + b} \Big)^a.$$

The following step in the order of business consists of minimizing (3.4) to obtain λ^* with which to construct Q^*. Once this has been carried out, according the prescriptions at the beginning of Section 2, the MEM reconstruction of ψ turns out to be

$$\psi_j^* = \frac{a}{((\mathbf{A}^*\lambda^*)_j + b)}. \tag{3.8}$$

We leave it up to the reader to double check that this time Q^* also happens to be a Γ distribution with different parameters, and that the components of ξ are not independent with respect to Q^*.

4 Numerical examples

This Section is devoted to analyzing a few of the many possibilities that may be dealt with. We shall begin with the simplest situation consisting of assuming that we are presented with the risk price of a liability which we known to have been priced coherently, but with a distortion function unknown to us. Recall that we are assuming as well that the distribution function of the risk is available to us, and obtaining it from the available data is the first step to be solved to implement our method. We shall consider a risk known to be distributed according to either a $U(0,1)$, a $Pareto(0,2)$, a $Gamma(1,2)$ or a $Beta(2,4)$ distribution. The computation of the risk price π of each liability was carried out with a distortion function of the (2.5) or (2.6) or (2.7) type. The parameters we use throughout are 1.5 for the proportional hazard and the dual power distortion function, and 0.05 for the Wang distortion function.

In Table 1 we indicate the reconstruction errors computed as $|\pi_k - (A\phi)_k^*|$, where ϕ^* is found as described in Section 3.2.1. For this we considered $p = q = 1/2$ and $a = 0; b = 6$.

Table 1: Reconstruction errors

Distortion	*Wang*		*PH*		*DP*	
	π	*error*	π	*error*	π	*error*
Uniform	0.51	0.577×10^{-10}	0.58	0.46×10^{-12}	0.4	0.29×10^{-10}
Pareto	3.96	0.91×10^{-10}	4.03	0.46×10^{-12}	3.0	0.09×10^{-10}
Gamma	2.69	0.81×10^{-11}	2.2	0.29×10^{-10}	1.55	0.14×10^{-11}
Beta	0.34	0.96×10^{-11}	0.34	0.43×10^{-10}	0.26	0.79×10^{-10}

In Figures 1 to 3 we plot the ϕ^*'s obtained using (3.6) and the identity $\Phi^* = \mathbf{C}\Psi^*$ for each spectral risk function, be it respectively, of Wang, proportional hazard or dual power types, as well as the original (true) ϕ itself. In each case the only datum was a the price of a different risk (either uniformly, Pareto, Beta or Gamma distributed) determined by the corresponding spectral risk aversion function. For example, in Figure 2, the dotted curve represents the reconstructed ϕ^* when the datum was the price of a $U(0,1)$ risk computed with a proportional hazard risk aversion function. We shall see below that the reconstruction improves as the number of risk prices taken into account increases. In this regard, the important thing is not that the reconstructed ϕ^*'s look like the true one, but that the reconstructed error is small. These ϕ^*'s can then be used to price other risks. In Table 2 we do the following comparison. We consider a $U(0,2)$ liability and compute is risk price according to the same three spectral risk functions as above, and we compare it with the risk price computed with the ϕ^* computed with the reconstructed spectral risk functions obtained above. For example, in the second row of the first column $\pi^* = 0.95$ denotes the price computed according to the discrete version of (2.9) where $X \sim U(0,2)$ and ϕ^* is the spectral risk function determined by the $Pareto(0,2)$ risk computed with the (2.7) spectral function.

Figure 1: Reconstructed ϕ's from one price determined by a Wang distortion

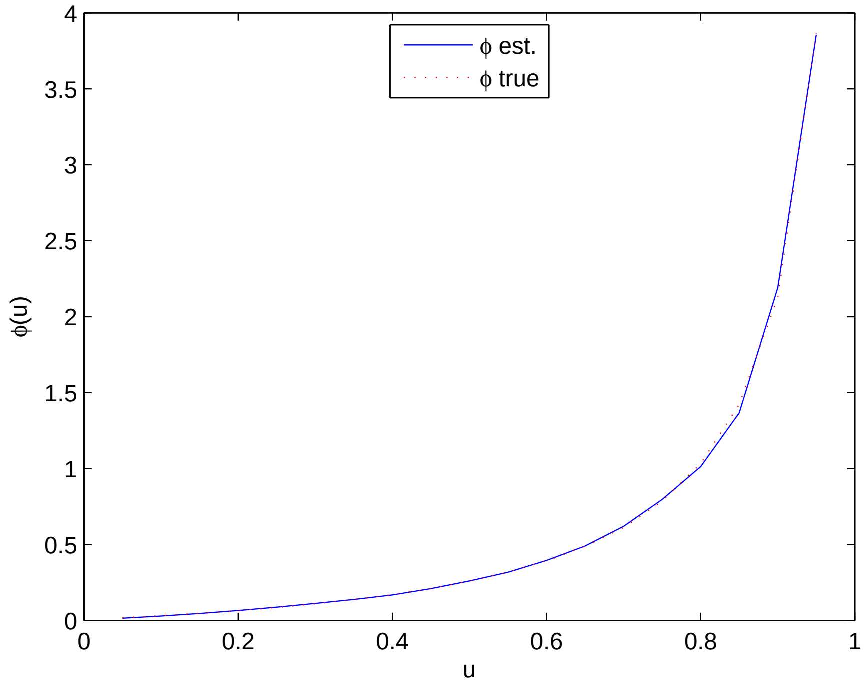


Figure 2: Reconstructed ϕ's from one price determined by a proportional hazard distortion

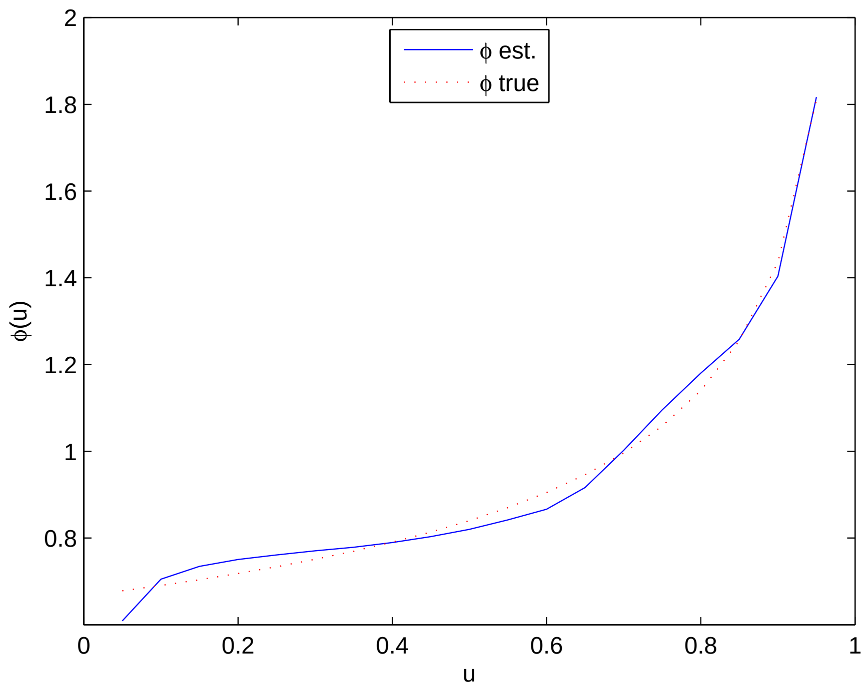
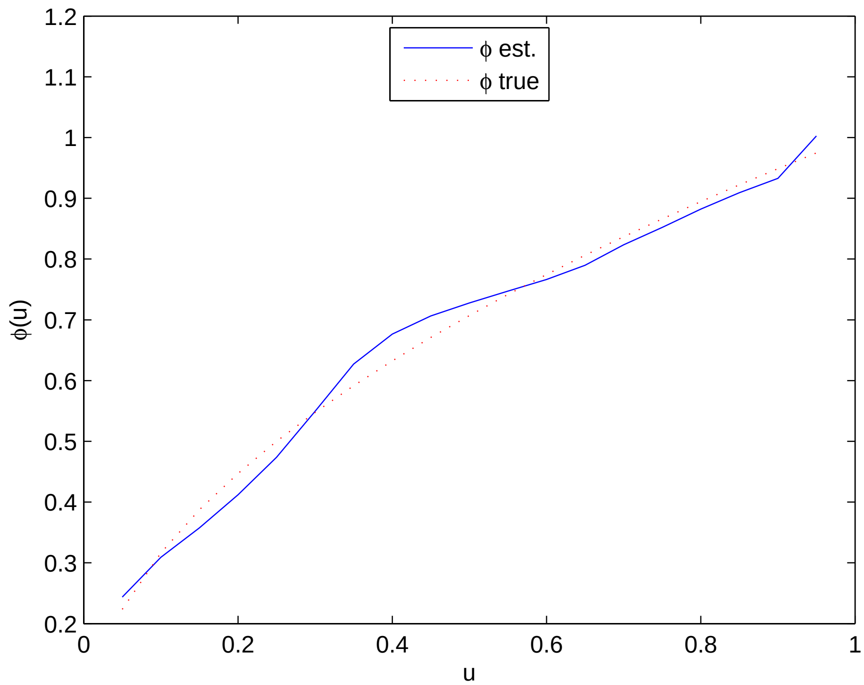


Figure 3: Reconstructed ϕ's from one price determined by a dual power distortion

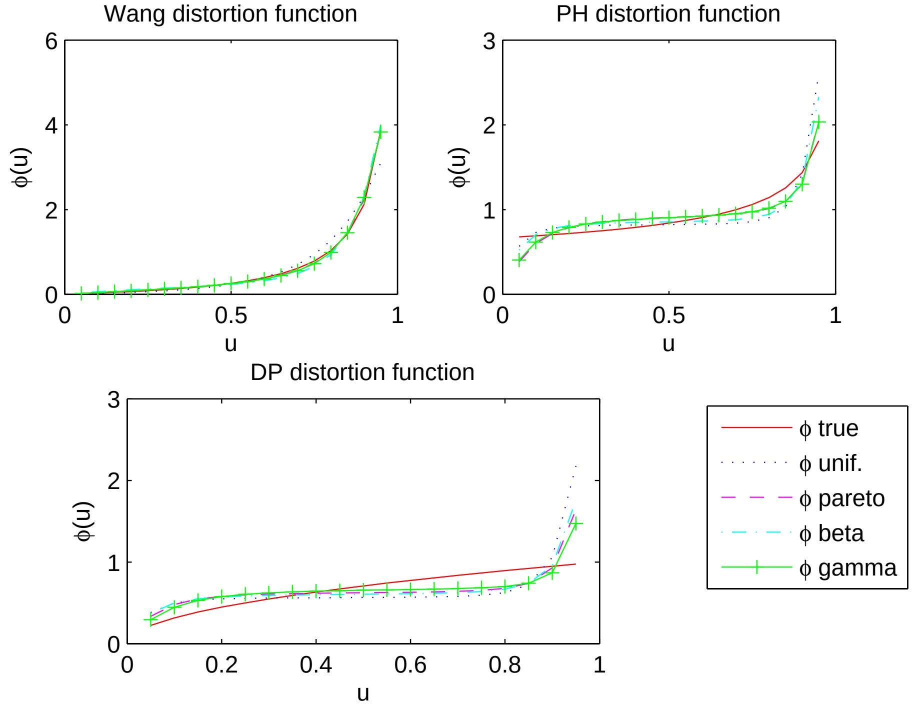


Distortion	*Wang*		*PH*		*DP*	
	π^*	*error*	π^*	*error*	π^*	*error*
Uniform	0.94	0.00039	1.02	0.0033	0.76	0.001
Pareto	0.95	0.003	1.03	0.00019	0.74	0.002
Gamma	0.98	0.002	1.05	0.002	0.72	0.003
Beta	0.96	0.003	1.03	0.004	0.77	0.002

Table 2: Error in risk price estimated with reconstructed ϕ^*.

In Table 3 we display the reconstruction error of each risk when the market prices of 4 liabilities are used to reconstruct one single spectral risk function. This time we considered a $U(0,1)$, a $Pareto(0,2)$, a $Gamma(2,4)$ and a $Beta(1,2)$, and the 4 liabilities were simultaneously priced with a (2.5), a (2.6) and a (2.7) risk aversion functions. Again the reconstructed ϕ^* was obtained with the method described in Section 3.2.1, with parameters $p = q = 1/2$ and $a =$ and $b = 6$. In Figure 4 we display the original spectral function ϕ and the reconstructed risk

Figure 4: Reconstructed ϕ's from four prices.



aversion function ϕ^*, when the price of one of four different liabilities, computed with one of the distortion functions was given as input. Surprisingly the method recognizes the distortion function regardless on the liability.

Table 3: Error of reconstruction risk price

Wang		*PH*		*DP*	
π	*error*	π	*error*	π	*error*
0.52	0.12×10^{-8}	0.55	0.15×10^{-5}	0.39	0.41×10^{-6}
4.01	0.001×10^{-8}	4.15	0.07×10^{-5}	2.79	0.012×10^{-6}
0.32	0.44×10^{-8}	0.35	0.34×10^{-5}	0.25	0.88×10^{-6}
2.53	0.02×10^{-8}	2.00	0.044×10^{-5}	1.60	0.07×10^{-6}

We compared the price of a $U(0, 2)$ liability computed with the same spectral risk aversion functions with the price computed with the reconstructed ϕ^*. The prices obtained, and absolute vales of the differences in price are: $\pi_{DP} = 0.7196$

and $|\pi_{DP} - \pi^*| = 0.0002$; $\pi_{Wang} = 1.008$ and $|\pi_{Wang} - \pi^*| = 2.11 \times 10^{-5}$ and $\pi_{PH} = 1.02$ with error $|\pi_{PH} - \pi^*| = 2.54 \times 10^{-6}$.

In table 4 we present results of a reconstruction process similar to those in table 3, but this time the procedure employed for the reconstruction is the one described in Section 3.2.2, where the parameters of the prior were $a = 3$ and $b = 1$. . The graphs of the reconstructed and original risk aversion functions are visually indistinguishable even though the reconstruction error are not small. This method is sensible when the ψ's can take arbitrarily large values.

Table 4: Reconstruction errors using the second method

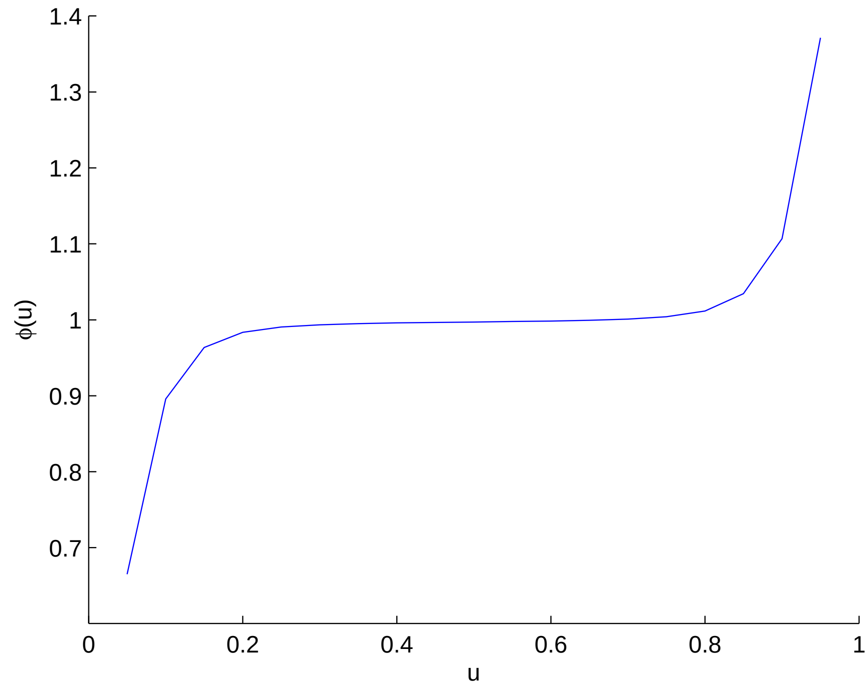
$Wang$	PH	DP
0.187×10^{-5}	0.089×10^{-3}	0.139×10^{-4}
0.009×10^{-5}	0.006×10^{-3}	0.002×10^{-4}
0.712×10^{-5}	0.173×10^{-3}	0.297×10^{-4}
0.025×10^{-5}	0.003×10^{-3}	0.016×10^{-4}

To finish, in Figure 5 we present the risk aversion function obtained when we consider a $U(0,1)$ liability and price it by the mean plus a small load. Actually we considered $\pi = (1 + 0.05)$. The reconstructed spectral function is constant except at the ends of the interval, meaning that the corresponding distorted function re weights only the small and the large probability events. But this flatness should not be interpreted according to the standard method of maximum entropy. Recall that we use the method of maximum entropy at each interval of the partition at which we reconstruct ϕ.

5 Concluding remarks

To sum up, when we are presented with the of prices of a collection of risks, all of them having been obtained with a common coherent risk measure described by a concave distortion function, it is possible to determine the distortion function

Figure 5: ϕ^* reconstructed from a loaded price



(or its associated risk aversion function) and to use it to consistently compute the price of other liabilities. When presented with the price of a single risk, it is possible to determine a risk aversion function that yields that price, and then use it for pricing other risks.

The method we propose, effective as it is, has to be applied with care. Not so much because of numerical issues, but because of modeling issues. The main reason being that the reconstruction depends on the choice of a prior constraint space and a prior measure Q_0, and of course, on the quality of the data. The first and the second issues are closely tied up, for the input data vector π must fall in the image $\mathbf{B}\{E_Q^*[\Phi] : \Phi \in \mathbf{K_0}\}$ of all possible spectral functions. In this regard the issues are that, on one hand, the data risk price vector π may not have been the result of a valuation process with a single distorted risk function, and on the other, market risk prices may deviate from their theoretical valuations. Both of these may make $\pi \notin \mathbf{B}\{E_Q^*[\Phi] : \Phi \in \mathbf{K_0}\}$. In this case the method of maximum entropy in the mean is not expected to produce an answer at all. In order to overcome these issues we shall provide an extension of the method in a forthcoming note.

The are no issues regarding the size of the partition. As the mesh becomes smaller, the approximation becomes better as shown in Gamboa and Gzyl (1997). The only thing to be kept in mind is not to include the extreme points of the interval $[0, 1]$ for there the distortion function may diverge to ∞.

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