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Pigouvian Second Degree Price Discrimination and  
Taxes in a Monopoly: an Example of Unit Tax  
Superiority

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ABSTRACT

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# Pigouvian second degree price discrimination and taxes in a monopoly: an example of unit tax superiority

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## Abstract

We provide an example of a monopoly with Pigouvian second-degree price discrimination where unit taxes are Pareto superior to *ad valorem* taxes.

*Keywords:* Monopoly, Pigouvian price discrimination, unit taxes, *ad valorem* taxes

JEL classification: D42, H21, L12

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## 1. Introduction

The comparison of *ad valorem* and unit taxation is an old matter in public economics. In competitive markets both taxation systems yield equal results, but they become dissimilar in non-competitive scenarios. The superiority of *ad valorem* taxes in monopoly settings was argued by Wicksell (1896) and proven by Suits and Musgrave (1953). Skeath and Trandel (1994) strengthen this result by showing that *ad valorem* taxes Pareto-dominate unit taxes, producing larger profits, tax revenues and consumer surplus.

The literature agrees that this superiority holds under monopoly in general. An example by Blackorby and Murty (2007) shows that, in a general equilibrium model with a monopoly sector, and in extreme case –one hundred percent profit taxation–, the *ad valorem* tax is not superior to the unit tax. Our example goes a step further showing Pareto superiority of the unit tax over the *ad valorem* tax, in a context of Pigouvian second-degree price discrimination. To the best of our knowledge this is the first example to provide such an insight for a monopoly. Such examples have been found already in some oligopoly models (see Anderson et al. (2001)).

We use a monopoly with Pigouvian second-degree price discrimination<sup>2</sup>. Although the modern view of second-degree price discrimination is focused on nonlinear pricing, we stick to the original, simpler definition. Our result is in

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<sup>2</sup>That is, the monopolist sets two different prices, such that all units with a demand price greater than the highest price ( $p_1$ ) are sold at  $p_1$ , while all units demanded at a price lower than  $p_1$  but higher than  $p_2$  are sold at  $p_2$  (see Pigou (1921)).

contrast with that of Cheung (1998), who generalizes the classical superiority of *ad valorem* taxes over unit taxes for first, second, and third degree price discrimination in a monopoly.

Pigouvian second-degree price discrimination is mainly of academic interest because it assumes myopic customers. But sales and markdowns are common events. And dynamic pricing has a long tradition in many industries, such as airlines or hotels. However, they often require complex models. Elmaghraby and Keskinocak (2003) provide several settings in which modeling customer behavior as myopic can be appropriate. Here, we limit ourselves to the theoretical study of Pigouvian price discrimination in a much simpler construct to illustrate a new possible result.

## 2. The Model

This section proceeds as follows. First, we will consider the case of a monopoly applying second price discrimination to a kinked demand. This type of demand is well known in economic theory; see for example Sweezy (1939) and Salop (1979). Secondly, we will investigate the effects of imposing a unit tax or an *ad valorem* tax. Then, we proceed to see that, under these conditions, the unit tax may be not only welfare superior but also Pareto superior to the *ad valorem* tax. Finally, we will illustrate our model with a numerical example.

Consider a monopoly with constant marginal cost  $c$  that applies second price discrimination. The firm faces a kinked linear demand such that:

$$p = \max \left\{ \min \left\{ a - bQ, r - sQ \right\}, 0 \right\},$$

with  $a, b, r, s > 0$ , and  $s > 4b$ .

The quantity at the bend is given by:

$$q = \frac{r - a}{s - b}. \quad (1)$$

In order to get a concave demand instead of a linear demand, we suppose that  $q > 0$  and  $\frac{r}{s} < \frac{a}{b}$ . For simplicity's sake, we do not impose differentiability of the function at point  $q$ .<sup>3</sup>

We will be concentrating on the cases where the monopolist's optimal choice is to select quantities in both segments of the demand. Let  $x$  denote the quantity produced by the monopolist to satisfy the first segment of the demand, while  $y$  represents the quantity sold at sale price. The monopolist will maximize its profit function, given by:

$$\Pi = (a - bx)x + (r - s(x + y))y - c(x + y).$$

So that the equilibrium quantities are:

$$x = \frac{r - 2a}{s - 4b} + \frac{c}{s - 4b}, \quad \text{and} \quad y = \frac{as - 2br}{s(s - 4b)} + \frac{c(2b - s)}{s(s - 4b)}. \quad (2)$$

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<sup>3</sup>Although this demand is not differentiable, our results do not depend substantially on the differentiability of the function. The results would hold if the demand function was to be substituted for the appropriate polynomial in the interval  $[q - \epsilon, q + \epsilon]$ .

The overall quantity produced by the monopolist is:

$$x + y = \frac{rs - as - 2br}{s(s - 4b)} + \frac{2bc}{s(s - 4b)}. \quad (3)$$

Note that, in our example, we assume that  $s > 4b$ , so that total quantity produced is increasing in  $c$ . But to simplify the analysis we will concentrate on the simplest case, where  $c = 0$ .

**Proposition 1.** *If  $c = 0$ , the solution presented in equation 2 holds if, and only if,  $\frac{3s}{s+2b} \leq \frac{r}{a} \leq \frac{s+2b}{3b}$ .*

PROOF. The necessary and sufficient conditions for this are: (i)  $0 \leq x$ ; (ii)  $x \leq q$ ; (iii)  $q \leq x + y$ ; and (iv)  $x + y \leq \frac{r}{s}$ .

We know that  $a, b, r, s > 0$ ; so  $s > 4b \Rightarrow s > b$ . Simple calculations show the following results:

(i) From equation 2,  $x \geq 0 \Leftrightarrow r/a \geq 2$ .

(ii) From equations 1 and 2,  $x \leq q$ , is equivalent to  $\frac{r}{a} \leq \frac{s+2b}{3b}$ .

(iii) From equations 1 and 3,  $x + y \geq q$ , can be expressed as  $\frac{r}{a} \geq \frac{3s}{s+2b}$ .

(iv) From equation 3,  $x + y \leq \frac{r}{s}$ , can be transformed to  $\frac{r}{a} \leq \frac{s}{2b}$ .

That is, when  $c = 0$ , the solution presented above holds if:

$$\max\left\{2, \frac{3s}{s+2b}\right\} \leq \frac{r}{a} \leq \min\left\{\frac{s+2b}{3b}, \frac{s}{2b}\right\}.$$

Since, if  $s > 4b$ , we have that  $2 < \frac{3s}{s+2b}$ , and  $\frac{s+2b}{3b} < \frac{s}{2b}$ . The solution holds if:

$$\frac{3s}{s+2b} \leq \frac{r}{a} \leq \frac{s+2b}{3b}. \quad (4)$$

Meaning that, for the purposes of our solution, (ii) and (iii) are the relevant inequalities<sup>4</sup>.

After describing the solution for a monopoly with second price discrimination, we study how the introduction of a tax affects the results. We first consider the introduction of an *ad valorem* tax,  $\tau$ . As it is well known, after the tax is put in place, the consumer will pay a price  $p$ , the firm will earn  $(1 - \tau)p$  and the government will collect  $\tau p$ . The perceived demand for the monopoly can be rewritten by multiplying every parameter by  $(1 - \tau)$  as:

$$p = \max\left\{0, \min\left\{a(1 - \tau) - b(1 - \tau)Q, r(1 - \tau) - s(1 - \tau)Q\right\}\right\}.$$

Faced with this demand, the monopolist will produce (as can be seen in equation 2 with  $c = 0$ ):

$$x_\tau = \frac{(1 - \tau)r - 2(1 - \tau)a}{(1 - \tau)s - 4(1 - \tau)b} = x, \quad \text{and} \quad y_\tau = \frac{(1 - \tau)^2 as - 2(1 - \tau)^2 br}{(1 - \tau)^2 s(s - 4b)} = y.$$

<sup>4</sup>It is not difficult to find examples that meet the condition expressed by equation 4, and one may serve at this point for clarification. Let  $b = 1$ , and  $s = 5$ , so that we get  $\frac{15}{7} < \frac{r}{a} < \frac{7}{3}$ .

This result implies that, for any meaningful solution, the monopolist will produce the same quantity with or without tax.

Let's now consider a unit tax,  $t$ . Now, for every unit the consumer pays  $p$ , the firm receives  $p - t$  and the government collects  $t$ . The perceived demand for the monopolist becomes:

$$p = \max \left\{ 0, \min \left\{ a - t - bQ, r - t - sQ \right\} \right\}.$$

According to equation 2, and maintaining the assumption of  $c = 0$ , a monopolist faced with this demand will produce

$$\begin{aligned} x_t &= \frac{r - t - 2(a - t)}{s - 4b}, \quad y_t = \frac{(a - t)s - 2b(r - t)}{s(s - 4b)} \\ x_t + y_t &= \frac{rs - as - 2br + 2bt}{s(s - 4b)}. \end{aligned} \quad (5)$$

We can now see that, when a unit tax is imposed and under the assumption  $s > 4b$ , a monopolist will increase both the output sold at the first (higher) price and the total production for any meaningful solution, since:

$$\frac{\partial x_t}{\partial t} = \frac{1}{s(s - 4b)} > 0, \quad \text{and} \quad \frac{\partial(x_t + y_t)}{\partial t} = \frac{2b}{s(s - 4b)} > 0.$$

If the increase in production is too large, the solution may become meaningless. We know, from Proposition 1 that a solution is meaningful whenever equation 4 holds. For any  $r > a > t > 0$  it is clear that  $\frac{r-t}{a-t} \geq \frac{r}{a}$ . In consequence, whenever  $\frac{r-t}{a-t} \leq \frac{s+2b}{3b}$ , the solution is meaningful. Or, expressed differently, whenever

$$t \leq \frac{a(s + 2b) - 3br}{s - b} = t_1 \quad (6)$$

We have come to the conclusion that the firm keeps production constant with any *ad valorem tax*, but production is increased under unitary taxation. This leads to the following proposition.

**Proposition 2.** *A unit tax may be welfare superior to an ad valorem tax if the monopolist engages in second degree price discrimination.*

Let's now compare how the government and the monopolist fare under both tax regimes. With *ad valorem* taxes, consumers pay

$$R_\tau = (a - bx)x + (r - s(x + y))y.$$

Substituting the quantities for their respective values from equations 2 and 3, this becomes:

$$R_\tau = \frac{as(r - a) - br^2}{s(s - 4b)}. \quad (7)$$

Which in turn means that government revenue will be  $G_\tau = \tau R_\tau$ , and the monopolist's profit is given by:  $\Pi_\tau = (1 - \tau)R_\tau$ .

With unit taxes, consumers pay

$$R_t = (a - bx_t)x_t + (r - s(x_t + y_t))y_t.$$

Using equation 5, this becomes:

$$R_t = \frac{as(r - a) + b(t^2 - r^2)}{s(s - 4b)}. \quad (8)$$

We can conclude that, as long as the solution is economically meaningful, consumers will pay more with a unit tax than with an *ad valorem* tax. For this reason, for any value of  $\tau$ , the additional income may be split between the monopolist and the government so both are better off.

Let us now turn to the consumer surplus and prove that consumers will be better off with a unit tax than with an *ad valorem* tax. Let  $V$  be the social surplus associated to a production level of  $x_\tau + y_\tau = x + y$ . Since production levels are increased under unitary taxation, social surplus will increase by an amount given by (the area of a trapezoid),

$$\Delta V(t) = \int_{x+y}^{x_t+y_t} (r - sz)dz = (x_t + y_t - x - y) \left( r - s \frac{x_t + y_t + x + y}{2} \right).$$

From equations 3 and 5,

$$x_t + y_t - x - y = \frac{2bt}{s(s - 4b)}, \text{ and } \frac{x_t + y_t + x + y}{2} = \frac{rs + bt - as - 2br}{s(s - 4b)}.$$

Consequently,

$$\Delta V(t) = \frac{2bt(as - bt - 2br)}{s(s - 4b)^2}.$$

The consumer surplus associated to the *ad valorem* tax is given by  $V - R_\tau$ , and by  $V + \Delta V(t) - R_t$  if taxes are unitary. The difference between the two, taking into account equations 7 and 8, is given by,

$$\Delta V(t) - R_t + R_\tau = \frac{bt(2as - 4br - (s - 2b)t)}{s(s - 4b)^2}.$$

This means that consumer surplus will be greater under unitary taxation as long as:

$$t \leq \frac{2as - 4br}{s - 2b} = t_2. \quad (9)$$

We will show that  $t_2 > t_1$ , as defined in equations 6 and 9. First it is clear that, as  $s > 4b$ ,

$$\frac{s - 2b}{s - b} = 1 - \frac{b}{s - b} > 1 - \frac{1}{3} = \frac{2}{3}.$$

In consequence, we have

$$t_2 - t_1 > \frac{2}{3} \frac{2as - 4br}{s - b} - \frac{a(s + 2b) - 3br}{s - b} = \frac{as + br - 6ab}{3(s - b)}.$$

Proposition 1 and  $s > 4b$ , imply that  $r/a \geq 2 > 6 - s/b$ . Multiplying by  $ab$ , and rearranging, yields  $as + br - 6ab > 0$ , and then,  $t_2 > t_1$ .

In consequence, with any meaningful solution (equation 6), consumer surplus would be higher under unitary taxation than under *ad valorem* taxation.

To guarantee equivalent tax yield for both tax regimes, we need  $G_t = G_\tau$ . The highest unit tax compatible with our solution is  $t_1$ . Let us call  $G_{t_1}$  to the government income with  $t_1$ . We define  $\tau_1$  as

$$\tau_1 = \frac{G_{t_1}}{R_\tau}. \quad (10)$$

Then for any  $\tau \leq \tau_1$ , we can find a unit tax  $t$ , such that  $t$  is Pareto superior to  $\tau$ . We can then conclude that<sup>5</sup>:

**Proposition 3.** *A unit tax may be Pareto superior to an ad valorem tax if the monopolist applies second degree price discrimination.*

### 2.1. Numerical example

To illustrate the propositions above, we consider now a numerical example with a demand function given by:

$$p = \max \left\{ \min \left\{ 9 - Q, 20 - 5Q \right\}, 0 \right\}$$

So that,  $a = 9$ ,  $b = 1$ ,  $r = 20$ , and  $s = 5$ . The monopoly solution, when the marginal cost is  $c = 0$ , would be  $q = 2.75$  and  $p = 6.25$ , resulting in a total income of 17.1875. However, monopoly profits would be higher under price discrimination.

As we saw before, an *ad valorem* tax would not change the quantities produced, so from equation 2 we get that  $x_\tau = 2$ ,  $y_\tau = 1$ . With the *ad valorem* tax,  $\tau$ , consumers pay a total of  $R_\tau = 19$ , the government collects  $G_\tau = 19\tau$ , and the firm gets a profit of  $\Pi_\tau = 19(1 - \tau)$ .

If a unitary tax was imposed instead, applying equation 5 we obtain  $x_t = 2 + t$ ,  $y_t = 1 - 0.6t$  and  $x_t + y_t = 3 + 0.4t$ . From equation 6, we can deduce that that maximum tax that may be imposed on the monopoly and still yield a meaningful result would be  $t = 0.75$ . Under this regime, consumers pay  $R_t = 19 + t^2/5$ , and, with any unitary tax of  $t \leq 0.75$ , government income would amount to  $G_t = 3t + 0.4t^2$ . The *ad valorem* tax that would guarantee an equivalent tax yield (according to equation 10) is  $\tau_1 = 0.13$ . Then for any *ad valorem* tax lower than 13%, a Pareto superior unit tax can be found.

We use Figure 2.1 to illustrate graphically the numerical example provided above. The figure on the right represents the situation of a monopoly with second price discrimination and a unitary tax ( $t = 0.75$ ), while the figure on the left represents the alternative *ad valorem* tax ( $\tau = 0.13$ ). In both cases, the areas shadowed in dark grey represent the profit accrued to the monopoly under each tax regime, while the areas in light gray represent the revenue of the government.

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<sup>5</sup>We developed this result based on the assumption that  $c = 0$ . As  $c$  increases, as can be seen in equation 3, the welfare advantage of unit taxes over *ad valorem* taxes decreases and may be reversed for high enough values of  $c$ .



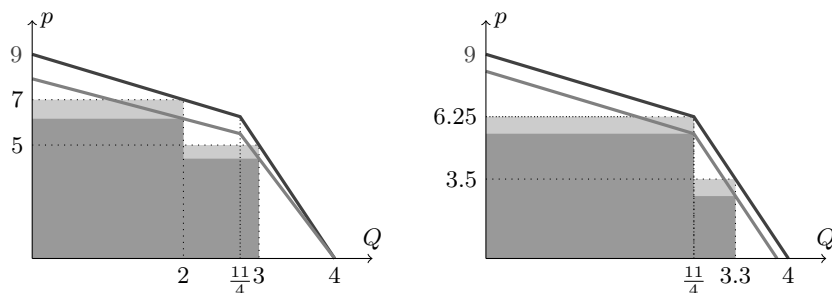


Figure 1: Pigouvian second degree price discrimination in a monopoly with unitary and *ad valorem* taxes

A tax of 13% may seem small, but other parameter values may result in higher tax rates for which unit taxes are superior. For example, with  $a = 10$ ,  $b = 1$ ,  $r = 25$  and  $s = 10$ , we know from equation 6 that  $t_1 = 5$ . Applying equation 5,  $x_{t_1} + y_{t_1} = \frac{11}{6}$ . And from equation 7,  $R_\tau = \frac{175}{12}$ . In consequence, from equation 10 the unit tax is *Pareto* superior to the *ad valorem* tax up to  $\tau_1 = \frac{22}{35}$  (that is, for any value of  $\tau$  between 0 and 62.8%).

### 3. Conclusion

In this paper we present an example where the generally accepted superiority of *ad valorem* taxes does not hold. We study a monopoly with Pigouvian second-degree price discrimination and find that, under certain conditions, unit taxes are not only welfare but also *Pareto* superior. This is not meant to be a general result contradicting accepted wisdom, but a theoretical example suggesting that the commonly held view deserves more careful consideration. In fact, it is possible that this result may be extended to other settings. On the one hand, the superiority of *ad valorem* taxes over unit taxes has long been established to be greater in monopoly than in other market structures. Given the result presented here for a monopoly, further investigation of the welfare effects of taxes in different settings may be interesting. Similarly, other types of non-Pigouvian second-degree price discrimination, such as nonlinear pricing, may be worth investigating.

Anderson, S. P., de Palma, A., Kreider, B., 2001. The efficiency of indirect taxes under imperfect competition. *Journal of Public Economics* 81 (2), 231 – 251.

Blackorby, C., Murty, S., 2007. Unit versus *ad valorem* taxes: Monopoly in general equilibrium. *Journal of Public Economics* 91 (34), 817 – 822.

Cheung, F. K., 1998. Excise taxes on a non-uniform pricing monopoly: *Ad valorem* and unit taxes compared. *The Canadian Journal of Economics / Revue canadienne d'Economique* 31 (5), pp. 1192–1203.

Elmaghraby, W., Keskinocak, P., 2003. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Science* 49 (10), pp. 1287–1309.

Pigou, A., 1921. *The economics of welfare*. MacMillan.

- Salop, S. C., 1979. Monopolistic competition with outside goods. *The Bell Journal of Economics* 10 (1), pp. 141–156.
- Skeath, S. E., Trandel, G. A., January 1994. A pareto comparison of ad valorem and unit taxes in noncompetitive environments. *Journal of Public Economics* 53 (1), 53–71.
- Suits, D. B., Musgrave, R. A., 1953. Ad valorem and unit taxes compared. *The Quarterly Journal of Economics* 67 (4), pp. 598–604.
- Sweezy, P. M., 1939. Demand under conditions of oligopoly. *Journal of Political Economy* 47, 568.
- Wicksell, K., 1896. Taxation in the monopoly case. In: Musgrave, R. A., Shoup, C. S. (Eds.), 1959. *Readings in the economics of taxation*. Allen & Unwin.