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Cross sections and pseudo-homomorphisms of topological abelian groups

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ABSTRACT

We say that a mapping ω between two topological abelian groups G and H is a pseudo-homomorphism if the associated map $(x, y) \in G \times G \mapsto \omega(x + y) - \omega(x) - \omega(y) \in H$ is continuous. This notion appears naturally in connection with cross sections (continuous right inverses for quotient mappings): given an algebraically splitting, closed subgroup H of a topological group X such that the projection $\pi : X \to X/H$ admits a cross section, one obtains a pseudo-homomorphism of X/H to H, and conversely. We show that H splits as a topological subgroup if and only if the corresponding pseudo-homomorphism can be decomposed as a sum of a homomorphism and a continuous mapping. We also prove that pseudo-homomorphisms between Polish groups satisfy the closed graph theorem. Several examples are given.

1. Introduction

Splitting problems for subgroups of topological groups have been studied for a long time. An ample collection of results within the class $\mathcal{L}$ of locally compact abelian groups appears in [1]. One of these results states that a topological group G in $\mathcal{L}$ splits in any group of $\mathcal{L}$ containing it if and only if G is topologically isomorphic to a product of copies of $\mathbb{R}$ and $\mathbb{T}$. Outside the locally compact case, the corresponding problem for subspaces of topological vector spaces has a long history, especially for Banach spaces, and it is still an object of active research.

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Within the general problem of finding conditions under which a closed subgroup H of a topological abelian group X splits, the particular case in which the corresponding quotient mapping $\pi : X \to X/H$ admits a cross section (i.e. a continuous right inverse mapping) is especially interesting. In this case there exists a homeomorphism between X and the product $H \times X/H$ leaving H invariant. If the cross section is additionally a homomorphism, then H splits as a topological subgroup.

We introduce and study the concept of pseudo-homomorphism, which is a natural strengthening of that of quasi-homomorphism as defined in [4]. A pseudo-homomorphism is a mapping $\omega : G \to H$ between topological abelian groups such that the associated mapping $(x, y) \in G \times G \mapsto \omega(x + y) - \omega(x) - \omega(y) \in H$ is continuous in $G \times G$. This notion appears in a natural way in connection with cross sections, thus providing an alternative framework for the above described problems. We define a distinguished class of pseudo-homomorphisms, namely those which can be decomposed as a sum of a homomorphism and a continuous function, and we prove that a pseudo-homomorphism is approximable in this sense precisely when the corresponding subgroup splits as a topological subgroup.

We devote the last section to prove that pseudo-homomorphisms between Polish groups satisfy the closed graph theorem.

1.1. Notation and preliminaries

All groups considered in this paper are abelian. We denote by $\mathcal{N}_0(X)$ the set of all neighborhoods of zero of the topological abelian group X.

We will sometimes write $H \leq X$ to indicate that H is a subgroup of the group X. We say that the subgroup H *splits algebraically* from X if there is a subgroup $H' \leq X$ such that the mapping $[(x, y) \in H \times H' \mapsto x + y \in X]$ is a group isomorphism. This is equivalent to the fact that there exists a group homomorphism $P : X \to H$ such that $P \circ i = \mathrm{id}_H$ where $i : H \to X$ is the inclusion mapping. We will refer to such a mapping P as an *algebraic retraction* for H in what follows. If $H \leq X$ is divisible, or if the quotient group X/H is free, then H splits algebraically from X.

If X is a topological group and H is a subgroup of X, we say that H *splits topologically* from X if there is a subgroup $H' \leq X$ such that $[(x, y) \in H \times H' \mapsto x + y \in X]$ is a topological isomorphism. It is clear that if H splits topologically from X then H, as well as any of its complements, is a closed subgroup of X. (But the converse is not true, see Proposition 18 below.) The closed subgroup H splits topologically from X if and only if there exists a continuous homomorphism $P : X \to H$ such that $P \circ i = \mathrm{id}_H$.

We denote by $\mathbb{T}$ the topological group $\mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ is endowed with the Euclidean topology. A *character* of an abelian group X is a homomorphism from X to $\mathbb{T}$. If X is a topological abelian group, we denote by $X^\wedge$ the dual group of X, which is defined as the group of all continuous characters of X with pointwise multiplication, endowed with the compact-open topology. If X is compact, $X^\wedge$ is discrete and vice-versa. By the classical Pontryagin–van Kampen theorem, every locally compact abelian group is canonically topologically isomorphic to its bidual group $(X^\wedge)^\wedge$.

For a completely regular Hausdorff space X, the free abelian topological group over X is the free abelian group $A(X)$ endowed with the unique Hausdorff group topology for which (1) the mapping $\eta : X \to A(X)$, which maps the topological space X onto a basis of $A(X)$, becomes a topological embedding and (2) for every continuous mapping $f : X \to G$, where G is an abelian Hausdorff group, there is a unique continuous group homomorphism $\widetilde{f} : A(X) \to G$ which satisfies $f = \widetilde{f} \circ \eta$.

A Polish space is a separable, completely metrizable space. By a classical result of Klee [13], if the topology of an abelian group can be generated by a complete metric, then it actually can be generated by a complete, invariant one. Hence a topological abelian group is Polish if and only if it is complete, separable and metrizable.

A k_ω-space is a Hausdorff topological space X which carries the weak topology with respect to an increasing sequence of compact subspaces whose union is X. Hausdorff locally compact σ-compact spaces are k_ω. Dual groups of metrizable groups are also k_ω.

For other topological notions we follow [8].

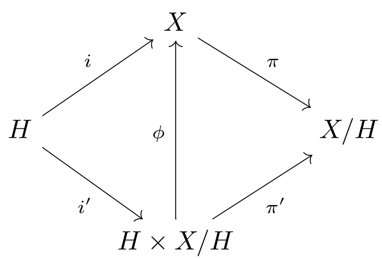
2. Cross sections, pseudo-homomorphisms and splitting subgroups

Definition 1. Let H be a closed subgroup of a topological abelian group X and let $\pi : X \to X/H$ be the canonical projection. A continuous map $s : X/H \to X$ is called a *cross section* for π, if $\pi \circ s = \mathrm{id}_{X/H}$.

Comfort, Hernández and Trigos-Arrieta studied cross sections for topological groups in [6] and they observed the following

Proposition 2. *Given a topological abelian group X, a closed subgroup H of X and the canonical projection $\pi : X \to X/H$, the following are equivalent:*

(1) *π has a cross section.*
(2) *There exists a retraction $r : X \to H$ such that $r(x+h) = r(x) + h$ for every $h \in H$ and $x \in X$.*
(3) *There exists a homeomorphism $\phi : H \times X/H \to X$ making the following diagram commutative:*



Proposition 3. *Let H be an open subgroup of a topological group X. The natural projection $\pi : X \to X/H$ has a cross section.*

Proof. Choose as $s : X/H \to X$ any map satisfying $s(\pi(x)) \in \pi^{-1}(x)$ for every $x \in X$. Since H is open, X/H is discrete and consequently s is continuous. $\square$

We denote by $G^\#$ an abelian group G endowed with its Bohr topology, i.e. the initial topology on G with respect to all homomorphisms from G to $\mathbb{T}$. For any subgroup H of G, the inclusion $H \to G$ is a topological embedding from $H^\#$ to $G^\#$ with closed image [5, Lemma 2.1 and p. 41]. Moreover the identity is a topological isomorphism between the topological abelian groups $G^\#/H^\#$ and $(G/H)^\#$ [14, Lemma 2.2].

In this context the following results are known (the symbol $\mathbb{Z}_p$ denotes the group of the p-adic integers):

Proposition 4.

(a) *([6, Theorem 24]) The projection $\mathbb{Q}^\# \to \mathbb{Q}^\#/\mathbb{Z}^\#$ has a cross section.*
(b) *([7, Example 3.9]) For every abelian group G such that $\mathbb{Z}_p \le G$, the projection $G^\# \to G^\#/\mathbb{Z}_p^\#$ has a cross section.*

The following Proposition provides another class of quotient maps admitting cross sections.

Proposition 5 *([2, Theorem 2.8]). Let X be a topological abelian group and H a compact subgroup of X. Assume that X/H is a zero-dimensional k_ω-space. Then the projection $\pi : X \to X/H$ admits a cross section.*

Topological vector spaces constitute a class of abelian topological groups when regarded under their additive structure. The following result, which appears as Proposition II.7.1 in [3], was originally proved by E. Michael:

Proposition 6. *Let E be a complete metric linear space and let L be a closed subspace of E. Assume that L is locally convex. Then the projection $\pi : E \to E/L$ has a cross section.*

We now introduce the notion of pseudo-homomorphism which, as we will see, is closely related with that of cross section.

Definition 7. Let G and H be topological Abelian groups and $\omega : G \to H$ a map with $\omega(0) = 0$. ω is called a *pseudo-homomorphism* if the map $\Delta_\omega : G \times G \to H$ defined by $\Delta_\omega(x, y) = \omega(x + y) - \omega(x) - \omega(y)$ is continuous.

Pseudo-homomorphisms can be characterized in the following way:

Lemma 8. *Let G and H be topological Abelian groups and $\omega : G \to H$ a map with $\omega(0) = 0$. Then ω is a pseudo-homomorphism if and only if it satisfies the following properties:*

(a) *The map $\Delta_\omega : (x, y) \in G \times G \mapsto \omega(x + y) - \omega(x) - \omega(y) \in H$ is continuous at $(0, 0)$.*
(b) *If the net (x_α) converges to $x \in G$ then $\omega(x_\alpha) - \omega(x_\alpha - x) \to \omega(x)$.*

Proof. If ω is a pseudo-homomorphism, (a) is trivially true. Let us prove (b): Let $x_\alpha \to x$ in G. From the continuity of Δ_ω it follows that $\Delta_\omega(x_\alpha, -x) \to \Delta_\omega(x, -x)$. Hence $\omega(x_\alpha - x) - \omega(x_\alpha) - \omega(-x) \to \omega(x - x) - \omega(x) - \omega(-x)$, i.e.

$$\omega(x_\alpha) - \omega(x_\alpha - x) \to \omega(x).$$

Conversely, assume that both (a) and (b) are true and pick two nets $(x_\alpha)_{\alpha\in A} \to x$ in G and $(y_\alpha)_{\alpha\in A} \to y$ in G. By hypothesis

$$\begin{aligned}
\omega(x_\alpha) - \omega(x_\alpha - x) &\to \omega(x),\\
\omega(y_\alpha) - \omega(y_\alpha - y) &\to \omega(y),\\
\omega(x_\alpha + y_\alpha) - \omega(x_\alpha + y_\alpha - x - y) &\to \omega(x + y),\\
\omega(x_\alpha - x + y_\alpha - y) - \omega(x_\alpha - x) - \omega(y_\alpha - y) &\to 0.
\end{aligned}$$

From continuity of the group operations it easily follows that

$$\omega(x_\alpha + y_\alpha) - \omega(x_\alpha) - \omega(y_\alpha) \to \omega(x + y) - \omega(x) - \omega(y).$$

Since x and y are arbitrary elements of G, we deduce that Δ_ω is continuous. □

Example 9. Let G and H be topological abelian groups. Let $f : G \to H$ be a continuous mapping such that $f(0) = 0$, and $a : G \to H$ a homomorphism. It is clear that $\omega = a + f$ is a pseudo-homomorphism.

This example serves as a motivation for the following definition:

Definition 10. A pseudo-homomorphism $\omega : G \to H$ is *approximable* if there exists a homomorphism $a : G \to H$ such that $\omega - a$ is continuous.

Proposition 11. *A pseudo-homomorphism $\omega : G \to H$ is approximable if there exists a homomorphism $a : G \to H$ such that $\omega - a$ is continuous at* 0*.*

Proof. Let us show that $\omega - a$ is actually continuous on G. Fix a net $(x_\alpha)_{\alpha \in A}$ in G which converges to $x \in G$. Since $\omega - a$ is a pseudo-homomorphism, by condition (b) in Lemma 8 we have $(\omega - a)(x_\alpha) - (\omega - a)(x_\alpha - x) \to (\omega - a)(x)$. Since $\omega - a$ is continuous at zero and $x_\alpha \to x$, we deduce $(\omega - a)(x_\alpha) \to (\omega - a)(x)$, as required. □

Let G and H be topological abelian groups and $\omega : G \to H$ a pseudo-homomorphism. As mentioned in [4, Lemma 2], it is not difficult to show that the family of sets $W(V, U) = \{(h, g) \in H \times G : g \in U, \ h \in \omega(g) + V\}$ where $U \in \mathcal{N}_0(G)$ and $V \in \mathcal{N}_0(H))$ is a basis of neighborhoods of zero for a group topology on $H \times G$. We will denote this group topology by τ_ω, and we will analyze it in terms of its convergent nets. They are characterized in the following Lemma:

Lemma 12. *With the above notations, a net (h_α, g_α) in $H \times G$ converges to (h, g) in the topology τ_ω if and only if $g_\alpha \to g$ in G and $h_\alpha - \omega(g_\alpha - g) \to h$ in H.*

Proof. Apply the definition of the topology τ_ω. □

The proof of the following Lemma is also immediate.

Lemma 13. *Let X and Y be topological groups. Let $\varphi : X \to Y$ be a homomorphism. Suppose that there exists a mapping $s : Y \to X$ such that $\varphi \circ s = id_Y$, $s(0) = 0$ and s is continuous at zero. Then φ is onto and open.*

We present in Theorems 14 and 17 a natural two-way relationship between cross sections and pseudo-homomorphisms. Under this correspondence, quotients by splitting subgroups are associated to approximable pseudo-homomorphisms.

Theorem 14. *Let G and H be topological abelian groups and $\omega : G \to H$ a pseudo-homomorphism. Let X be the topological group $H \times G$, endowed with the group topology τ_ω. Then the natural inclusion of H into X is an embedding, and the natural projection $\pi : X \to G$ is a quotient mapping which admits a cross section. Moreover, ω is approximable if and only if H is a splitting subgroup of X.*

Proof. From Lemma 12 it follows at once that for any net (h_α) in H, one has $h_\alpha \to 0$ in H if and only if $(h_\alpha, 0) \to (0, 0)$ in X. This implies that H is a topological subgroup of X.

Consider the mapping $s : G \to X$ given by $s(g) = (\omega(g), g)$. It is clear that $\pi \circ s = \mathrm{id}_G$. Let us see that s is continuous. By Lemma 12, we need to show that for every $g \in G$ and every net (g_α) with $g_\alpha \to g$, we have $\omega(g_\alpha) - \omega(g_\alpha - g) \to \omega(g)$. This follows from condition (b) in Lemma 8.

Again from Lemma 12 it is immediate to deduce that π is continuous. Using Lemma 13 we conclude that it is a quotient mapping.

Now assume that ω is approximable, say $\omega = a + f$ where $a : G \to H$ is a homomorphism and $f : G \to H$ is continuous. Define $P : X \to H$ as $P(h, g) = h - a(g)$. This is clearly an algebraic retraction for H. Let us show that it is continuous, which will imply that H is a splitting subgroup of X. Fix any net (h_α, g_α) in $H \times G$ which converges to $(0, 0)$ in τ_ω. This implies by Lemma 12 that $g_\alpha \to 0$ and $h_\alpha - \omega(g_\alpha) \to 0$. Since $\omega = a + f$ and f is continuous, we deduce $h_\alpha - a(g_\alpha) = P(h_\alpha, g_\alpha) \to 0$.

Conversely, assume that H is a splitting subgroup of X. Let $P : X \to H$ be a continuous homomorphism with $P \circ i = \mathrm{id}_H$. Define $a(g) = -P(0, g)$ for any $g \in G$; let us see that $\omega - a : G \to H$ is continuous at zero, which by Proposition 11 will imply that ω is approximable. Fix a net (g_α) which converges to 0 in G. Then $\omega(g_\alpha) - a(g_\alpha) = \omega(g_\alpha) + P(0, g_\alpha) = P(\omega(g_\alpha), 0) + P(0, g_\alpha) = P(\omega(g_\alpha), g_\alpha) = P(s(g_\alpha)) \to 0$. □

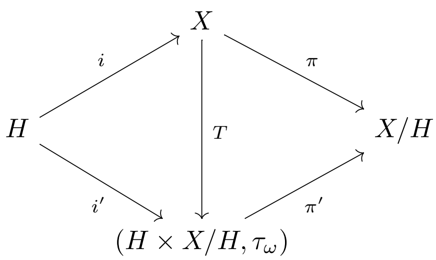
Corollary 15. *Let $A(X)$ be the free abelian topological group on a Tychonoff topological space X and let H be any topological abelian group. Then every pseudo-homomorphism $\omega : A(X) \to H$ is approximable.*

Proof. According to Theorem 14 the natural projection $\pi : (H \times A(X), \tau_\omega) \to A(X)$ is a quotient mapping which admits a continuous cross section $s : A(X) \to (H \times A(X), \tau_\omega)$. The restriction $s|_X : X \to (H \times A(X), \tau_\omega)$ is clearly continuous, hence there exists a continuous homomorphism $S : A(X) \to (H \times A(X), \tau_\omega)$ such that $S|_X = s|_X$. It is clear that $\pi \circ S = \mathrm{id}_{A(X)}$, which implies that H is a splitting subgroup of $(H \times A(X), \tau_\omega)$ and, again by Theorem 14, ω is approximable. □

Proposition 16. *Let G be the product of a family of locally precompact abelian groups and let H be a product of copies of $\mathbb{R}$ and/or $\mathbb{T}$. Then any pseudo-homomorphism $\omega : G \to H$ is approximable.*

Proof. By Corollary 3.14 in [2], H is a splitting subgroup of $(H \times G, \tau_\omega)$ and by Theorem 14, ω is approximable. □

Theorem 17. *Let H be a closed subgroup of a topological abelian group X which splits algebraically and let $P : X \to H$ be an algebraic retraction. If the canonical projection $\pi : X \to X/H$ admits a cross section s with $s(0) = 0$ then $\omega = P \circ s$ is a pseudo-homomorphism and there is a topological isomorphism T making the following diagram commutative:*



In particular ω is approximable if and only if H splits topologically from X.

Proof. Let us see that Δ_ω is continuous. Since $\pi \circ s = \mathrm{id}_{X/H}$, for every $g_1, g_2 \in X/H$ we have $\pi(s(g_1 + g_2) - s(g_1) - s(g_2)) = 0$ which implies that $s(g_1 + g_2) - s(g_1) - s(g_2) \in H$. We deduce $\omega(g_1 + g_2) - \omega(g_1) - \omega(g_2) = P(s(g_1 + g_2) - s(g_1) - s(g_2)) = s(g_1 + g_2) - s(g_1) - s(g_2)$. Since s is continuous, the result follows.

Define now $T(x) = (P(x), \pi(x))$ for every $x \in X$. Clearly T makes the above diagram commutative. For every net (x_α) in X we have

$$\begin{aligned} T(x_\alpha) \to (0,0) &\Leftrightarrow \pi(x_\alpha) \to 0 \quad \text{and} \quad P(x_\alpha - s(\pi(x_\alpha))) \to 0 \quad \text{(by Lemma 12)} \\ &\Leftrightarrow \pi(x_\alpha) \to 0 \quad \text{and} \quad x_\alpha - s(\pi(x_\alpha)) \to 0 \quad (\text{because } x_\alpha - s(\pi(x_\alpha)) \in H) \\ &\Leftrightarrow x_\alpha \to 0. \end{aligned}$$

Finally, it is easy to check that $(h, g) \mapsto s(g) + h - P(s(g))$ is the inverse of T.

The fact that ω is approximable if and only if H splits topologically from X follows from the commutativity of the above diagram and the corresponding result for $(H \times X/H, \tau_\omega)$ (Theorem 14). □

We devote the remaining of this section to prove a few results showing the ubiquity of non-approximable pseudo-homomorphisms. In the first one we will use some well-known structural and duality-related properties of locally compact abelian groups; proofs of those results can be found in [10].

Proposition 18. *For every compact, connected abelian group H which is not topologically isomorphic to a product of copies of $\mathbb{T}$ there exists a compact, totally disconnected abelian group G and a non-approximable pseudo-homomorphism $\omega : G \to H$.*

Proof. The abelian group $H^\wedge$ is not free, hence there is an abelian group A and a (algebraically) non-splitting torsion subgroup $T \leq A$ with $A/T \cong H^\wedge$ (Corollary 3.2 in [9]). By Pontryagin duality, H is canonically a non-splitting subgroup of the compact group $A^\wedge$ and $A^\wedge/H \cong T^\wedge$ [10, 23.25, 24.11]. Since H is compact and connected, it is divisible, hence H splits algebraically from $A^\wedge$. Moreover, $A^\wedge/H$ is totally disconnected, hence zero-dimensional. By Proposition 5, the projection $A^\wedge \to A^\wedge/H$ admits a cross section. By Theorem 17, the associated pseudo-homomorphism cannot be approximable. □

A subspace L of a topological vector space E is said to be *complemented* if there is a subspace L' of E such that the linear mapping $[(x, y) \in L \times L' \mapsto x + y \in E]$ is a topological isomorphism. It is easy to see that a subspace $L \leq E$ is complemented in E if and only if it splits topologically as a subgroup of E.

Proposition 19. *Let E be a complete metric linear space and let L be a non-complemented, locally convex, closed subspace of E. Then there exists a non-approximable pseudo-homomorphism $\omega : E/L \to L$.*

Proof. By Proposition 6 the projection $\pi : E \to E/L$ has a cross section. It is clear that L splits algebraically from E. Since L does not split topologically as a subgroup of E, by Theorem 17, we deduce that there exists a non-approximable pseudo-homomorphism $\omega : E/L \to L$. □

Example 20. Let ℓ^1 be the classical Banach space of all summable real sequences $x = (x_n)$ endowed with the norm $\|x\| = \sum |x_n|$. There is a non-approximable pseudo-homomorphism $\omega : \ell^1 \to \mathbb{R}$. Indeed, it is well-known that there exists a complete metric linear space E with a non-complemented, one-dimensional subspace L such that E/L is topologically isomorphic, as a topological vector space, to the Banach space ℓ^1 (see [12, Ch. 5.4]). The existence of a non-approximable pseudo-homomorphism $\omega : \ell^1 \to \mathbb{R}$ follows from Proposition 19. This example yields the following remarkable fact: The topological group $(\mathbb{R} \times \ell^1, \tau_\omega)$ is homeomorphic to $\mathbb{R} \times \ell^1$ (by Proposition 2) and $\mathbb{R} \times \{0\}$ splits algebraically from $(\mathbb{R} \times \ell^1, \tau_\omega)$, but it does not split topologically.

3. A closed graph theorem for pseudo-homomorphisms

It is well known that any homomorphism between Polish groups which has a closed graph is continuous. From here it is not difficult to deduce that any approximable pseudo-homomorphism between Polish groups which has a closed graph is also continuous. In this section we will see that this property is true without the approximability assumption.

The following result can be regarded as a version of Theorem 1.1 in [15] for topological abelian groups.

Theorem 21. *Let G and H be Polish abelian groups and let $f : G \to H$ be a mapping which satisfies the following conditions:*

(a) *If $x_n \to 0$, $y_n \to 0$, $f(x_n) \to 0$ and $f(y_n) \to 0$ then $f(x_n + y_n) \to 0$.*
(b) *If $x_n \to 0$ and $f(x_n) \to 0$ then $f(-x_n) \to 0$.*
(c) *If $x_n \to x$ then $[f(x_n - x) \to 0 \Leftrightarrow f(x_n) \to f(x)]$.*

If the graph of f is closed in $G \times H$ then f is continuous.

Proof. Note that (c) implies that $f(0) = 0$. Let $\Gamma \subset G \times H$ be the graph of f. Define the mapping $\star : \Gamma \times \Gamma \to \Gamma$ by $(x, f(x)) \star (u, f(u)) = (x+u, f(x+u))$. Then $(\Gamma, \star)$ is an abelian group where $(0, f(0)) = (0, 0)$ is the additive identity and the additive inverse of $(x, f(x))$ is $(-x, f(-x))$. Let us see that it is actually a topological group with the topology on Γ induced by the product topology on $G \times H$.

Suppose that $(x_n, f(x_n)) \to (x, f(x))$ and $(y_n, f(y_n)) \to (y, f(y))$. Let us see that $(x_n+y_n, f(x_n+y_n)) \to (x + y, f(x + y))$. It is clear that $x_n + y_n \to x + y$, so we only have to prove $f(x_n + y_n) \to f(x + y)$. By hypothesis $x_n \to x$ and $f(x_n) \to f(x)$; hence by condition (c) we get $f(x_n - x) \to 0$. In the same way we deduce $f(y_n - y) \to 0$. By applying condition (a) to the sequences $x_n - x$ and $y_n - y$ we get $f((x_n + y_n) - (x + y)) \to 0$. Since $x_n + y_n \to x + y$, again by condition (c) we deduce $f(x_n + y_n) \to f(x + y)$.

Now suppose that $(x_n, f(x_n)) \to (x, f(x))$ and let us see that $(-x_n, f(-x_n)) \to (-x, f(-x))$. Since $x_n \to x$ and $f(x_n) \to f(x)$, by condition (c) we have that $f(x_n - x) \to 0$. Applying (b) to the sequence $x_n - x$ we deduce $f(-x_n + x) \to 0$. Since $-x_n \to -x$, again from (c) we get $f(-x_n) \to f(-x)$.

Since Γ is a subspace of the separable, metrizable space $G \times H$, it is separable itself. We conclude that Γ is a Polish group. The projection $p : (x, f(x)) \in \Gamma \mapsto x \in G$ is a continuous algebraic isomorphism. By applying the corresponding Open Mapping Theorem [11, Corollary 32.4] to p we conclude that p^{-1} is continuous and thus f is continuous, too. □

Corollary 22. *Let G and H be Polish abelian groups and let $\omega : G \to H$ be a pseudo-homomorphism. If the graph of ω is closed in $G \times H$ then ω is continuous.*

Proof. Since Δ_ω is continuous at $(0, 0)$, it is clear that ω satisfies conditions (a) and (b) in Theorem 21. Condition (c) is a consequence of Lemma 8(b). The assertion follows. □

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