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One-parameter subgroups of topological abelian groups ☆

M.J. Chasco

Dept. de Física y Matemática Aplicada, University of Navarra, Spain

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ABSTRACT

It is proved that, for a wide class of topological abelian groups (locally quasi-convex groups for which the canonical evaluation from the group into its Pontryagin bidual group is onto) the arc-component of the group is exactly the union of the one-parameter subgroups. We also prove that for metrizable separable locally arc-connected reflexive groups, the exponential map from the Lie algebra into the group is open.

Some properties of one-parameter subgroups of locally compact groups have been known for a long time. In a paper published in 1967 Rickert proved that in a compact arc-connected group every point lies on a one-parameter subgroup [16]. Previously Gleason had shown in 1950 that every finite dimensional, locally compact group contains a one-parameter subgroup [10]. There are also examples of topological groups without nontrivial one-parameter subgroups; this is the case for instance of the subgroup of integer-valued functions of the Hilbert space $L^2[0,1]$ [1].

For topological abelian groups which are k-spaces, the arc-component of the dual group is the union of its one-parameter subgroups. This result published by Nickolas in 1977, remained for some years the only available piece of information outside the class of locally compact groups. Recently it was proved in [1] that the same is true for a much wider class of topological groups.

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E-mail address: mjchasco@unav.es.

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In this paper we go deeper into the study of one-parameter subgroups of topological abelian groups. We take as references the papers [1,2,15] and the book by Hofmann and Morris [12] which presents a wide range of properties of the one-parameter subgroups of locally compact abelian groups.

In order to do that we use two ingredients: on the one hand Pontryagin duality techniques and on the other the relation between $\mathrm{CHom}(\mathbb{R}, G)$ and the group G given by the evaluation mapping

$$\mathrm{CHom}(\mathbb{R}, G) \longrightarrow G, \ \varphi \longmapsto \varphi(1)$$

The vector space $\mathrm{CHom}(\mathbb{R}, G)$ endowed with the compact open topology is called the Lie algebra of the topological group G and denoted by $\mathcal{L}(G)$ in analogy with the classical theory of Lie groups. In that case the evaluation mapping is continuous and it is called the exponential function ($\exp_G$). The elements of $\operatorname{im}\exp_G$ are those lying on one-parameter subgroups, and G is the union of its one-parameter subgroups if and only if $\exp_G$ is onto.

For a topological group G, we denote by G_a its arc-component.

The main results of the paper are:

Theorem 6. *Let G be a Hausdorff locally quasi-convex topological abelian group for which the evaluation mapping from the group into its bidual group is onto. Then,* $\operatorname{im}\exp_G = G_a$.

Theorem 11. *If G is a metrizable separable reflexive topological abelian group which has an arc-connected neighborhood of e_G, then the exponential mapping* $\exp_G : \mathcal{L}(G) \longrightarrow G_a$ *is a quotient map.*

Observe that if G is the additive topological group underlying a topological vector space E, then $\exp_G : \mathrm{CHom}(\mathbb{R}, G) \longrightarrow G, \ \varphi \longmapsto \varphi(1)$ is a topological isomorphism and in that case all the topological and algebraic information about G is in $\mathrm{CHom}(\mathbb{R}, G)$.

We will use Pontryagin duality theory, hence our topological groups need to be abelian. Pontryagin–van Kampen duality theorem for locally compact abelian groups (LCA groups) is a very deep result. It is the base of Abstract Harmonic Analysis and it allows to know the structure of LCA groups. This explains why a consistent Pontryagin–van Kampen duality theory has been developed for general topological abelian groups and why the abelian topological groups satisfying the Pontryagin–van Kampen duality, the so-called *reflexive groups*, have received considerable attention from the late 40's of the last century (see [8] for a survey on the subject).

For an abelian topological group G, the *character group* or *dual group* $G^\wedge$ of G is defined by

$$G^\wedge := \{\chi : G \to \mathbb{T} : \ \chi \text{ is a continuous homomorphism}\}$$

where $\mathbb{T}$ denotes the compact group of complex numbers of modulus 1. The elements of $G^\wedge$ are called *characters*. We say that the group G *has enough characters* or that G is a *MAP* group if for any $e_G \neq x \in G$, there is some character $\varphi \in G^\wedge$ such that $\varphi(x) \neq 1$.

Endowed with the compact-open topology $G^\wedge$ is an abelian Hausdorff group. The *bidual* group $G^{\wedge\wedge}$ is $(G^\wedge)^\wedge$ and the canonical mapping in the bidual group is defined by:

$$\alpha_G : G \to G^{\wedge\wedge}, \ x \mapsto \alpha_G(x) : \varphi \mapsto \varphi(x).$$

The group G is called *Pontryagin reflexive* if α_G is a topological isomorphism. For the sake of simplicity, we will use the term *reflexive* only. The famous Pontryagin–van Kampen theorem states that every locally compact abelian (LCA) group is reflexive.

Let $f : G \to E$ be a continuous homomorphism of topological groups. The *dual mapping* $f^\wedge : E^\wedge \to G^\wedge$ defined by $(f^\wedge(\chi))(g) := (\chi \circ f)(g)$ is a continuous homomorphism.

The *annihilator of* a subgroup $H \subset G$ is defined as the subgroup $H^{\perp} := \{\varphi \in G^{\wedge}: \varphi(H) = \{1\}\}$. If L is a subgroup of $G^{\wedge}$, the *inverse annihilator* is defined by $L^{\perp} := \{g \in G: \varphi(g) = 1,\ \forall \varphi \in L\}$.

Annihilators are the particularizations for subgroups of the more general notion of polars of subsets. Namely, for $A \subset G$ and $B \subset G^{\wedge}$, the polar of A is $A^{\triangleright} := \{\varphi \in G^{\wedge} : \varphi(A) \subset \mathbb{T}_+\}$ and the inverse polar of B is $B^{\triangleleft} := \{g \in G: \varphi(g) \in \mathbb{T}_+,\ \forall \varphi \in B\}$, where $\mathbb{T}_+ := \{z \in \mathbb{T} : \operatorname{Re} z \geq 0\}$.

For a topological abelian group G, it is not difficult to prove that a set $M \subset G^{\wedge}$ is *equicontinuous* if there exists a neighborhood U of the neutral element in G such that $M \subset U^{\triangleright}$. Other standard facts in duality theory are: If U is a neighborhood of the neutral element of G, its polar is compact and the dual group $G^{\wedge}$ is the union of all polars of neighborhoods of e_G. The family $\{K^{\triangleright} : K \text{ is a compact subset of } G\}$ is a neighborhood basis of the neutral element in $G^{\wedge}$.

A subgroup H of a topological group G is said to be *dually closed* if, for every element x of $G \setminus H$, there is a continuous character φ in $G^{\wedge}$ such that $\varphi(H) = \{1\}$ and $\varphi(x) \neq 1$.

Reflexive groups lie in a wider class of groups, the so-called *locally quasi-convex groups*. Quasi-convexity was defined by Vilenkin as a sort of convexity for abelian topological groups inspired on the Hahn–Banach theorem for locally convex spaces.

A subset A of a topological group G is called *quasi-convex* if for every $g \in G \setminus A$, there is some $\varphi \in A^{\triangleright}$ such that $\varphi(g) \notin \mathbb{T}_+$.

It is easy to prove that for any subset A of a topological group G, $A^{\triangleright\triangleleft}$ is a quasi-convex set. It will be called the *quasi-convex hull* of A since it is the smallest quasi-convex set that contains A. Obviously, A is quasi-convex iff $A^{\triangleright\triangleleft} = A$.

If A is a subgroup of G, A is quasi-convex if and only if A is dually closed. The abelian topological group G is said to be locally quasi-convex if it admits a neighborhood basis of e_G formed by quasi-convex subsets. Dual groups are examples of locally quasi-convex groups. For locally quasi-convex groups the evaluation map α_G is injective and open onto its image. A topological vector space E is locally convex if and only if in its additive structure it is a locally quasi-convex topological group (see [6, 2.4]).

By a *real character* (as opposed to a *character*) on an abelian topological group G it is commonly understood a continuous homomorphism from G into the reals $\mathbb{R}$. The real characters on G constitute the vector space $\mathrm{CHom}(G, \mathbb{R})$. It is said that the group G has *enough real characters* if $\mathrm{CHom}(G, \mathbb{R})$ separates the points of G. We denote by $\mathrm{CHom}_{co}(G, \mathbb{R})$ the group of real characters endowed with the compact open topology. We say that a character $\varphi : G \to \mathbb{T}$ can be *lifted to a real character*, if there exists a real character f such that $e^{2\pi i f} = \varphi$. We denote by $G^{\wedge}_{\mathrm{lift}}$ the subgroup of $G^{\wedge}$ formed by the characters that can be lifted to a continuous real character.

Given a topological abelian group G, we denote by $\omega(G, G^{\wedge})$ the weak topology on G that is, the topology on G induced by the elements of $G^{\wedge}$. This topology coincides with the Bohr topology.

On the other hand $\omega(G^{\wedge}, G)$ denotes the topology on $G^{\wedge}$ of pointwise convergence.

In a topological group G, the arc-components are homeomorphic to one another and it makes sense to refer to the arc-component of the neutral element G_a as the arc-component.

There are well known results about the arc-component of locally compact groups. Let G be an LCA group, G_a be its arc-component and G_0 be its connected component.

1. $G_a = \operatorname{im} \exp_G$ (see [12, 8.30]).
2. The arc-component G_a is dense in the connected component G_0 (see [12, 7.71]).
3. The group G has enough real characters iff the dual group $G^{\wedge}$ is connected (see [11, 24.35]).

What can be said about the arc-component in more general classes of groups? It is known that for topological abelian groups which are k-spaces, the arc-component of the dual group is exactly the union of its one-parameter subgroups (see [15]). It was proved recently that the same is true for a much wider class of topological groups: groups satisfying the EAP condition. (A topological group G satisfies the EAP if every arc in $G^\wedge$ is equicontinuous.) This property is introduced and studied for different groups in [1].

Next, we find new classes of groups for which $G_a = \operatorname{im}\exp_G$.

We start with a lemma that can be proved in a straightforward way.

Lemma 1. *Let G be a Hausdorff topological abelian group, H be a subgroup of G and L a subgroup of $G^\wedge$ then,*

H is dense in the weak topology $\omega(G, G^\wedge)$ iff $H^\perp = \{e_{G^\wedge}\}$.

L is dense in the pointwise topology $\omega(G^\wedge, G)$ if $L^\perp = \{e_G\}$. If G is a MAP group, the reverse implication is also true.

Lemma 2. *Let G be a Hausdorff topological abelian group. If K is a compact subset of G, then $\alpha_G(K)$ is an equicontinuous subset of $G^{\wedge\wedge}$.*

Proof. The polar set $K^\triangleright$ is a neighborhood of $e_{G^\wedge}$ for the compact open topology of $G^\wedge$, then $K^{\triangleright\triangleright}$ is an equicontinuous subset of $G^{\wedge\wedge}$. Moreover, $\alpha_G(K^{\triangleright\triangleleft}) = K^{\triangleright\triangleright} \bigcap \alpha_G(G)$. Therefore $\alpha_G(K) \subseteq \alpha_G(K^{\triangleright\triangleleft}) = K^{\triangleright\triangleright} \bigcap \alpha_G(G) \subseteq K^{\triangleright\triangleright}$. Since $\alpha_G(K)$ is contained in an equicontinuous subset, it is itself equicontinuous. □

Proposition 3. *Let G be a Hausdorff topological abelian group and $\gamma : \mathbb{I} \to G$ be a continuous arc, then the mapping $\Phi_\gamma : G^\wedge \times \mathbb{I} \to \mathbb{T}$ defined by $\Phi_\gamma(\chi, t) = \chi(\gamma(t))$ is continuous.*

Proof. Take $\chi_0 \in G^\wedge$ and $t_0 \in \mathbb{I}$. Let us see that Φ_γ is continuous at (χ_0, t_0). For $n \in \mathbb{N}$, let us denote by $\mathbb{T}_n$ the neighborhood of 1 in $\mathbb{T}$, $\{e^{2\pi i t} : |t| \leq \frac{1}{4n}\}$. Fix $n \in \mathbb{N}$; since the mapping $\chi_0 \circ \gamma : \mathbb{I} \to \mathbb{T}$ is continuous, there exists V_{t_0} neighborhood of t_0 in $\mathbb{I}$ such that $\chi_0(\gamma(t))\overline{\chi_0(\gamma(t_0))} \in \mathbb{T}_{2n}$, for every $t \in V_{t_0}$.

On the other hand since $\gamma(\mathbb{I})$ is a compact subset of G, by Lemma 2, $\alpha_G(\gamma(\mathbb{I}))$ is equicontinuous at χ_0 hence, there exists a neighborhood U_{χ_0} of χ_0 in $G^\wedge$ such that $\chi(\gamma(t))\overline{\chi_0(\gamma(t))} \in \mathbb{T}_{2n}$ for every $t \in \mathbb{I}$ and $\chi \in U_{\chi_0}$.

Therefore $\chi(\gamma(t))\overline{\chi_0(\gamma(t))}\chi_0(\gamma(t))\overline{\chi_0(\gamma(t_0))} \in \mathbb{T}_{2n}\mathbb{T}_{2n} \subset \mathbb{T}_n$ for every $t \in V_{t_0}$ and $\chi \in U_{\chi_0}$. This proves that Φ_γ is continuous at (χ_0, t_0). □

Proposition 4. *Let G be a Hausdorff topological abelian group and G_a be its arc-component, then $G_a \leq \alpha_G^{-1}(G^{\wedge\wedge}_{\mathrm{lift}})$.*

Proof. Let x be an element in G_a, and let $\gamma : \mathbb{I} \to G$ be a continuous mapping joining e_G and x. Then, $\Phi_\gamma : G^\wedge \times \mathbb{I} \to \mathbb{T}$ defined by $\Phi_\gamma(\chi, t) = \chi(\gamma(t))$ is continuous as we have seen in Proposition 3. Denote by $\psi : G^\wedge \times \{0\} \to \mathbb{R}$ the null real character. By the homotopy lifting property we can find a homotopy $F : G^\wedge \times \mathbb{I} \to \mathbb{R}$ such that $p \circ F = \Phi_\gamma$ and $F\,|_{G^\wedge \times \{0\}} = \psi$. Now the unique path lifting property of $p : \mathbb{R} \to \mathbb{T}$ allows us to show that $\varphi : G^\wedge \to \mathbb{R}$ defined as the restriction of F to $G^\wedge \times \{1\}$ is a homomorphism and hence a continuous real character lifting $\alpha_G(x)$: for all $l \in G^\wedge$, $p \circ \varphi(l) = p \circ F(l, 1) = \Phi_\gamma(l, 1) = l(\gamma(1)) = l(x) = \alpha_G(x)(l)$. □

Proposition 5. *Let G be a Hausdorff locally quasi-convex topological abelian group for which α_G is onto. Then, $\alpha_G^{-1}(G^{\wedge\wedge}_{\mathrm{lift}}) \leq \operatorname{im}\exp_G \leq G_a$.*

Proof. Observe that $\operatorname{im}\exp_G$ is arc-connected and so it is contained in G_a. Let $x \in G$ be such that $\alpha_G(x) \in G^{\wedge\wedge}_{\mathrm{lift}}$ and let $\widetilde{\alpha_G(x)} : G^\wedge \to \mathbb{R}$ be a continuous homomorphism such that $p \circ \widetilde{\alpha_G(x)} = \alpha_G(x)$. Denote by S the topological isomorphism $S : \mathbb{R} \to \mathbb{R}^\wedge$, $s \to \chi_s : \chi_s(t) = e^{2\pi i s t}$. Since G is locally quasi-convex, the evaluation mapping α_G is injective and open, so we can consider the homomorphism $f \in \mathrm{CHom}(\mathbb{R}, G)$ given by $f = \alpha_G^{-1} \circ (\widetilde{\alpha_G(x)})^\wedge \circ S$. Let us see that $f(1) = x$. Observe first that $(\widetilde{\alpha_G(x)})^\wedge(S(1)) = \alpha_G(x)$ [for

$\psi \in G^\wedge$: $(\widetilde{\alpha_G(x)})^\wedge(S(1))(\psi) = (\chi_1 \circ \widetilde{\alpha_G(x)})(\psi) = e^{2\pi i \widetilde{\alpha_G(x)(\psi)}} = (p \circ \widetilde{\alpha_G(x)})(\psi) = \alpha_G(x)(\psi)]$. Therefore, $f(1) = \alpha_G^{-1} \circ (\widetilde{\alpha_G(x)})^\wedge(S(1)) = \alpha_G^{-1}(\alpha_G(x)) = x$. $\square$

The following theorem, obtained from the previous propositions, shows in particular that in an arc-connected locally quasi-convex topological abelian group for which α_G is onto, every point lies in a one-parameter subgroup. The same was proved by Rickert in 1967 for compact arc-connected groups (see [16]).

Theorem 6. *Let G be a Hausdorff locally quasi-convex topological abelian group for which α_G is onto. Then,* $\operatorname{im}\exp_G = G_a = \alpha_G^{-1}(G^{\wedge\wedge}_{\text{lift}})$.

Some classes of Hausdorff locally quasi-convex topological abelian groups for which α_G is onto are the following: reflexive groups, duals of pseudocompact groups, P-groups (see [8]), groups of continuous functions $C(X, \mathbb{T})$ where X is a completely regular k-space (see [3, 14.8]) and the wide class of nuclear complete topological abelian groups [3, 21.5].

The class of nuclear groups was formally introduced by Banaszczyk in [6]. It is a class of topological groups embracing nuclear spaces and locally compact abelian groups (as natural generalizations of finite-dimensional vector spaces). The definition of nuclear groups is very technical, as could be expected from its virtue of joining together objects of such different classes. A nice survey on nuclear groups is also provided by L. Außenhofer in [4]. The following are important facts concerning the class of nuclear groups:

1. Nuclear groups are locally quasi-convex, [6, 8.5].
2. Products, subgroups and quotients of nuclear groups are again nuclear, [6, 7.5].
3. Every locally compact abelian group is nuclear, [6, 7.10].
4. Every closed subgroup of a nuclear topological group is dually closed [6, 8.6].
5. A nuclear locally convex space is a nuclear group, [6, 7.4]. Furthermore, if a topological vector space E is a nuclear group, then it is a locally convex nuclear space, [6, 8.9].

Proposition 7. *([5, 1.4]) Let G be a Hausdorff locally quasi-convex topological abelian group for which α_G is continuous, then the mapping $\Phi_0 : \mathcal{L}(G) \to \mathrm{CHom}_{co}(G^\wedge, \mathbb{R}^\wedge)$ given by $\varphi \mapsto \varphi^\wedge$: $\varphi^\wedge(\chi) = \chi \circ \varphi$ is an embedding.*

Proposition 8. *Let G be a reflexive topological abelian group, then the mapping $\Phi_0 : \mathcal{L}(G) \to \mathrm{CHom}_{co}(G^\wedge, \mathbb{R}^\wedge)$ given by $\varphi \mapsto \varphi^\wedge$ is a topological isomorphism.*

Proof. Using Proposition 7 we only need to check that Φ_0 is onto: Let $\psi : G^\wedge \to \mathbb{R}^\wedge$ be a continuous homomorphism and $\varphi = \alpha_G^{-1} \circ \psi^\wedge \circ \alpha_{\mathbb{R}}$. Let us see that $\varphi^\wedge = \psi$ or which is the same, that for every $\chi \in G^\wedge$, $\varphi^\wedge(\chi) = \chi \circ \varphi = \chi \circ \alpha_G^{-1} \circ \psi^\wedge \circ \alpha_{\mathbb{R}} = \psi(\chi)$.

So, take $t \in \mathbb{R}$, if $x = (\alpha_G^{-1} \circ \psi^\wedge \circ \alpha_{\mathbb{R}})(t)$, then $\alpha_G(x) = \alpha_{\mathbb{R}}(t) \circ \psi$

$$\chi(x) = \alpha_G(x)(\chi) = (\alpha_{\mathbb{R}}(t) \circ \psi)(\chi) = \alpha_{\mathbb{R}}(t)(\psi(\chi)) = \psi(\chi)(t).$$

Therefore $\varphi^\wedge(\chi)(t) = \psi(\chi)(t)$, for all $\chi \in G^\wedge$ and $t \in \mathbb{R}$, that is $\varphi^\wedge = \psi$. $\square$

Corollary 9. *Let G be a Hausdorff locally quasi-convex topological abelian group for which α_G is continuous, then $\mathcal{L}(G)$ is topologically isomorphic to a subgroup of $\mathrm{CHom}_{co}(G^\wedge, \mathbb{R})$. Moreover, if G is reflexive then $\mathcal{L}(G) \cong \mathrm{CHom}_{co}(G^\wedge, \mathbb{R})$.*

Proof. Take into account that $S : \mathbb{R} \to \mathbb{R}^\wedge$ given by $S(t)(s) = e^{2\pi i st}$ is a topological isomorphism. $\square$

We recall now that some properties of the group G preserved by $\mathcal{L}(G)$ are

1. $\mathcal{L}(G)$ is Hausdorff, if the topological group G is Hausdorff.
2. $\mathcal{L}(G)$ is complete, if the topological group G is complete.
3. $\mathcal{L}(G)$ is a locally convex space, if the topological group G is a Hausdorff locally quasi-convex group (see [5, 1.2]).
4. $\mathcal{L}(G)$ is nuclear if the topological group G is nuclear (see [5, 2.6]).

Proposition 10. *If G is a locally quasi-convex metrizable and separable topological abelian group, $\mathrm{CHom}_{co}(G^\wedge, \mathbb{R})$ is metrizable complete and separable. The Lie algebra $\mathcal{L}(G)$ is also metrizable and separable, and it is complete if the topological group G is complete.*

Proof. Since G is metrizable, $G^\wedge$ is hemicompact and a k-space [7]. By the hemicompactness of $G^\wedge$, $\mathrm{CHom}_{co}(G^\wedge, \mathbb{R})$ is metrizable and because $G^\wedge$ is a k-space, $\mathrm{CHom}_{co}(G^\wedge, \mathbb{R})$ is complete. On the other hand since G is metrizable and separable it holds (see [9, 1.7]) that compact subsets of $G^\wedge$ are metrizable. But for a hemicompact k-space X whose compact subsets are metrizable, $C_{co}(X)$ is metrizable and separable (see [14] and [17]), hence $\mathrm{CHom}_{co}(G^\wedge, \mathbb{R})$ is metrizable separable and complete. Observe now that $\mathcal{L}(G)$ is complete because G is complete and it is metrizable and separable because it is topologically isomorphic to a subgroup of $\mathrm{CHom}_{co}(G^\wedge, \mathbb{R})$. $\square$

Remark. For torus groups $\mathbb{T}^X$ where X is an arbitrary set, we may identify the exponential function with the canonical quotient map $\mathbb{R}^X \to \mathbb{T}^X$, therefore, it is open. There are also compact connected groups which are not torus groups but for which the exponential function is open onto its image (see [13]). We next find other classes of abelian groups for which the corestriction $\exp_G : \mathcal{L}(G) \longrightarrow G_a$ is open.

Theorem 11. *If G is a metrizable separable reflexive topological abelian group which has an arc-connected neighborhood of e_G, then the exponential mapping $\exp_G : \mathcal{L}(G) \longrightarrow G_a$ is a quotient map.*

Proof. By the above proposition, $\mathcal{L}(G)$ is metrizable complete and separable. On the other hand, since G is a locally quasi-convex topological group for which α_G is onto, $\operatorname{im} \exp_G = G_a$. By hypothesis, G_a is an open subgroup of the group G, hence it is closed. Therefore G_a is metrizable complete. Since the exponential mapping is a continuous and onto homomorphism, by the open mapping theorem, it is open. $\square$

Corollary 12. *Let G be a metrizable separable reflexive topological group:*

1. *If G is locally arc-connected, then the exponential map $\exp_G : \mathcal{L}(G) \longrightarrow G_a$ is a quotient map.*
2. *If G is arc-connected, then the exponential map $\exp_G : \mathcal{L}(G) \longrightarrow G$ is a quotient map.*
3. *If the exponential map $\exp_G : \mathcal{L}(G) \longrightarrow G_a$ is a quotient map and G has an arc-connected neighborhood of e_G, then G is locally arc-connected.*

Proof. 3. Since $\mathcal{L}(G)$ is a locally convex topological vector space it has arbitrarily small arc-connected neighborhoods of zero which are mapped onto open identity neighborhoods of G_a by $\exp_G$. $\square$

The following lemma shows that for a topological abelian group, to have enough real characters and to have enough characters that can be lifted, are equivalent properties.

Lemma 13. *([2, 2.1]) Let G be a topological abelian group and $e_G \neq x \in G$. The following assertions are equivalent:*

1. *There exists a real character f such that $f(x) \neq 0$.*
2. *There exists a character $\varphi \in G^{\wedge}_{\text{lift}}$ such that $\varphi(x) \neq 1$.*

Corollary 14. *A Hausdorff topological abelian group G has enough real characters if and only if $(G^{\wedge}_{\text{lift}})^{\perp} = \{e_G\}$.*

Theorem 15. *([1]) Let (G, τ) be an abelian Hausdorff satisfying EAP. Then*

$$G^{\wedge}_{\text{lift}} = \operatorname{im} \exp_{G^{\wedge}} = (G^{\wedge})_a$$

Theorem 16. *Let (G, τ) be an abelian Hausdorff group such that every arc in the character group is equicontinuous. Consider the following conditions:*

a) G has enough real characters.
b) $(G^{\wedge})_a$ is dense in $G^{\wedge}$, with the pointwise convergence topology.

Then a) $\Rightarrow$ b). If G is a MAP group, b) $\Rightarrow$ a).

Proof. By Theorem 15, $((G^{\wedge})_a)^{\perp} = (G^{\wedge}_{\text{lift}})^{\perp}$. If the group G has enough real characters $(G^{\wedge}_{\text{lift}})^{\perp}$ is trivial and by Lemma 1, $(G^{\wedge})_a$ is dense in $G^{\wedge}$ with the pointwise convergence topology. For the reverse implication take into account that for a MAP group G, $(G^{\wedge})_a$ dense in $G^{\wedge}$ with the pointwise convergence topology, implies $((G^{\wedge})_a)^{\perp}$ is trivial. □

Corollary 17. *Let G be a locally quasi-convex topological abelian group such that α_G is onto then,*

1. *G_a is $\omega(G, G^{\wedge})$-dense iff $G^{\wedge}$ has enough real characters.*
2. *G is arc-connected iff every character of $G^{\wedge}$ can be lifted.*

Proof. (1) Since $G_a = \alpha_G^{-1}(G^{\wedge\wedge}_{\text{lift}})$ we have $(G_a)^{\perp} = (\alpha_G^{-1}(G^{\wedge\wedge}_{\text{lift}}))^{\perp} = (G^{\wedge\wedge}_{\text{lift}})^{\perp}$. By Lemma 1, G_a is $\omega(G, G^{\wedge})$-dense iff $(G_a)^{\perp} = \{e_{G^{\wedge}}\}$ and by Corollary 14, $(G^{\wedge\wedge}_{\text{lift}})^{\perp} = \{e_{G^{\wedge}}\}$ iff $G^{\wedge}$ has enough real characters then: G_a is $\omega(G, G^{\wedge})$-dense iff $G^{\wedge}$ has enough real characters.

(2) Again by Theorem 6, $G_a = G$ iff $G^{\wedge\wedge}_{\text{lift}} = G^{\wedge\wedge}$. □

Corollary 18. *Let G be a locally quasi-convex topological abelian group with α_G onto and such that closed subgroups are dually closed. If $G^{\wedge}$ has enough real characters, then G is connected.*

Proof. Having $G^{\wedge}$ enough real characters the arc-component G_a of G is $\omega(G, G^{\wedge})$-dense. Let G_0 be the connected component of G. It is clear that $G_a \subset G_0$ therefore G_0 is $\omega(G, G^{\wedge})$-dense. Since G_0 is a closed subgroup of the group G, G_0 is dually closed and therefore it is $\omega(G, G^{\wedge})$-closed. Then $G_0 = G$, that is: G is connected. □

Examples. Nuclear complete topological abelian groups are locally quasi-convex topological abelian groups with α_G onto and such that closed subgroups are dually closed. Therefore every nuclear complete topological group, such that its dual group, $G^{\wedge}$ has enough real characters, is connected.

The previous results allows us to give an alternative proof for the following well known fact.

Corollary 19. *Let G be an LCA group. Then G is connected iff $G^{\wedge}$ has enough real characters.*

Proof. Observe that LCA groups are nuclear complete, so the if part is true. Let us prove the reverse implication. Since G is an LCA group, G_a is dense in G_0. But $G_0 = G$ because G is connected, so G_a is dense and therefore $\omega(G, G^\wedge)$-dense in G. Therefore, by Corollary 17, $G^\wedge$ has enough real characters. □

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