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Permutable fuzzy consequence and interior operators and their connection with fuzzy relations

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ABSTRACT

Fuzzy operators are an essential tool in many fields and the operation of composition is often needed. In general, composition is not a commutative operation. However, it is very useful to have operators for which the order of composition does not affect the result. In this paper, we analyze when permutability appears. That is, when the order of application of the operators does not change the outcome. We characterize permutability in the case of the composition of fuzzy consequence operators and the dual case of fuzzy interior operators. We prove that for these cases, permutability is completely connected to the preservation of the operator type.

We also study the particular case of fuzzy operators induced by fuzzy relations through Zadeh's compositional rule and the inf-$\rightarrow$ composition. For this cases, we connect permutability of the fuzzy relations (using the sup-$*$ composition) with permutability of the induced operators. Special attention is paid to the cases of operators induced by fuzzy preorders and similarities. Finally, we use these results to relate the operator induced by the transitive closure of the composition of two reflexive fuzzy relations with the closure of the operator this composition induces.

1. Introduction

Fuzzy consequence operators and fuzzy interior operators are an essential tool in most of the different frameworks were fuzzy logic appears. As prominent examples, we find approximate reasoning and fuzzy mathematical morphology. In approximate reasoning, fuzzy consequence operators are used to obtain conclusions from certain fuzzy premises and fuzzy relations [13,18,19,30]. Fuzzy interior operators appear as a dual notion of fuzzy consequence operators in the lattice of truth values [5]. In fuzzy mathematical morphology, fuzzy consequence operators and fuzzy interior operators are called fuzzy closings and openings respectively and they act as morphological filters used for image processing [7,8,15,16]. Operators induced by fuzzy relations appear in this context as a generalization of morphological filters defined in sets were an additive operation does not necessarily exist [20,22]. In these cases, the fuzzy relation plays the role of structuring element. This abstraction allows to use certain techniques from fuzzy mathematical morphology into data mining problems [21]. Other places where fuzzy consequence and interior operators appear are modal logic [10], fuzzy topology [23–25], fuzzy rough sets [9,28,34], fuzzy relation equations [29] and fuzzy concept analysis [1,3].

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In all these contexts there is a need of concatenating two or more operators and it is important to know when this composition preserves their properties. A very relevant question is whether the composition of two fuzzy consequence (interior) operators is such an operator. As will be shown it this paper, it turns out to be closely related to their permutability or commutativity.

The objective of this paper is to characterize permutability in the case of composition of either two fuzzy consequence operators or two fuzzy interior operators. We study two particular cases of operators induced by fuzzy relations: fuzzy operators induced by means of Zadeh compositional rule and fuzzy operators induced by the inf-$\rightarrow$ composition, mainly focusing on the cases of operators induced by fuzzy preorders and similarities. As we shall see, permutability of fuzzy relations is closely related to permutability of their induced fuzzy operators and preservation of their properties.

The paper is organized as follows. In Section 2 we set the framework and we recall the main definitions and results that will be used throughout the paper.

In Section 3 we recollect several definitions and results that show connections between fuzzy relations and fuzzy operators. We recall the operators C_R^* and $C_R^{\rightarrow}$ given by Zadeh's compositional rule and inf-$\rightarrow$ composition respectively, several of their properties and extend the notion of C_R^* to a more general process to obtain fuzzy operators from a fuzzy relation and another fuzzy operator. We introduce the notion of concordance between a fuzzy operator and a fuzzy relation, which is the key to preserve the properties of fuzzy consequence operator of the induced operator.

Sections 4 and 5 are devoted to the analysis of permutability for certain cases of fuzzy relations and fuzzy operators. In Section 4, we study permutability of general fuzzy preorders and the particular case of fuzzy indistinguishability relations. In Section 5, permutability of fuzzy consequence operators is characterized and dual results are obtained for the case of fuzzy interior operators.

Sections 6 and 7 show the relationship between permutability of fuzzy relations and permutability of fuzzy operators by using the connections established in Section 3. In Section 6, we relate permutability of fuzzy relations with permutability of the operators that they induce through Zadeh's compositional rule. In Section 7, a similar study is made for operators induced through inf-$\rightarrow$ composition. We use the results developed in Sections 4 and 5 in order to study the cases of fuzzy operators induced by fuzzy preorders and similarities.

In Section 8 we analyze under which conditions different properties of the induced operators are satisfied even if permutability does not hold. Some of these properties are used to relate the operator induced by the transitive closure of the composition of two reflexive fuzzy relations with the closure of the operator this composition induces.

Finally, in Section 9 we present the conclusions.

2. Preliminaries

Let X be a non-empty classical set and let $[0,1]^X$ denote the set of all fuzzy subsets of X with truth values in $[0,1]$ endowed with the structure of complete commutative residuated lattice (in the sense of Bělohlávek [4]). That is, $\langle [0,1], \wedge, \vee, *, \rightarrow, 0, 1\rangle$ where $\wedge$ and $\vee$ are the usual infimum and supremum, $*$ is a left-continuous t-norm and $\rightarrow$ is the residuum of $*$ defined for $\forall a, b \in X$ as $a \rightarrow b = \sup\{\gamma \in [0,1] | a * \gamma \leqslant b\}$.

Recall that $*$ and $\rightarrow$ satisfy the adjoiness property

$$x * y \leqslant z \Longleftrightarrow y \leqslant x \rightarrow z$$

and that $*$ is monotone in both arguments while $\rightarrow$ is antitone in the first argument and monotone in the second one.

As always, the inclusion of fuzzy sets is defined by the pointwise order, i.e. $\mu \subseteq \nu$ if and only if $\mu(x) \leqslant \nu(x)$ for all $x \in X$.

Let us recall some properties of $\langle [0,1], \wedge, \vee, *, \rightarrow, 0, 1\rangle$ that will be used in the paper. Detailed proofs can be found in [4].

Proposition 2.1. *The residuated lattice* $\langle [0,1], \wedge, \vee, *, \rightarrow, 0, 1\rangle$ *satisfies the following conditions for each index set I and for all* $x, x_i, y, y_i, z \in [0,1]$ *with* $i \in I$:

1. $1 \rightarrow x = x$	6. $(x \rightarrow y) * (y \rightarrow z) \leqslant (x \rightarrow z)$
2. $x \leqslant y \Longleftrightarrow x \rightarrow y = 1$	7. $x * \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x * y_i)$
3. $x * 0 = 0$	8. $x \rightarrow \bigwedge_{i \in I} y_i = \bigwedge_{i \in I}(x \rightarrow y_i)$
4. $x * (x \rightarrow y) \leqslant y$	9. $\bigvee_{i \in I} x_i \rightarrow y = \bigwedge_{i \in I}(x_i \rightarrow y)$
5. $(x * y) \rightarrow z = x \rightarrow (y \rightarrow z)$	$x * \bigwedge_{i \in I} y_i \leqslant \bigwedge_{i \in I}(x * y_i)$

We will use the notation sup or $\vee$ for the supremum and inf or $\wedge$ for the infimum indistinctly.

Recall that every partially ordered set P, and therefore every lattice, gives rise to a dual (or opposite) partially ordered set which usually denoted P^δ. P^δ is defined to be the set P with the inverse order, i.e. $x \leqslant y$ holds in P^δ if and only if $y \leqslant x$ holds in P. It is easy to see that this construction allows us to translate every statement from P to a statement P^δ by replacing each occurrence of $\leqslant$ by $\geqslant$. Notice that if P is a lattice, every occurrence of $\vee$ gets replaced by $\wedge$ and vice versa [14].

A fuzzy operator is a map $C : [0,1]^X \longrightarrow [0,1]^X$. We denote Ω' the set of all fuzzy operators on the referential set X. Recall that Ω' is a lattice with order given by $C \leqslant C'$ if and only if $C(\mu) \subseteq C'(\mu)$ for all $\mu \in [0,1]^X$. All the operations are pointwise inherited from the structure given to $[0,1]$.

Definition 2.1. A fuzzy operator $C \in \Omega'$ is called a fuzzy consequence operator or fuzzy closure operator (FCO for short) when it satisfies for all $\mu, \nu \in [0,1]^X$:

(C1) Inclusion $\mu \subseteq C(\mu)$
(C2) Monotonicity $\mu \subseteq \nu \Rightarrow C(\mu) \subseteq C(\nu)$
(C3) Idempotence $C(C(\mu)) = C(\mu)$

Ω will denote the set of all fuzzy consequence operators of $[0,1]^X$.

Definition 2.2. A fuzzy operator $C \in \Omega'$ is called a fuzzy interior operator (FIO for short) when it satisfies for all $\mu, \nu \in [0,1]^X$:

(I1) Anti-inclusion $C(\mu) \subseteq \mu$
(I2) Monotonicity $\mu \subseteq \nu \Rightarrow C(\mu) \subseteq C(\nu)$
(I3) Idempotence $C(C(\mu)) = C(\mu)$

Λ will denote the set of all fuzzy interior operators of $[0,1]^X$.

Fuzzy consequence operators were introduced by Pavelka in 1979 as an extension of Tarski's consequence operators to fuzzy sets [30]. In approximate reasoning, they perform the role of deriving consequences from certain premises and relations [19,30,13,18]. From an algebraic point of view, fuzzy consequence operators are the closure operators the lattice $[0,1]^X$ [35]. Fuzzy interior operators appear as a dual notion of fuzzy closure operators [5]. They can be seen as fuzzy consequence operators in the dual lattice Ω'^{δ}. One can prove that Ω' and Ω'^{δ} are isomorphic through the function $\varphi : \Omega' \longrightarrow \Omega'^{\delta}$ defined as $\varphi(C) = 1 - C$ where $1 - C$ is the fuzzy operator defined as $(1 - C)(\mu)(x) = 1 - C(\mu)(x)$ for every $\mu \in [0,1]^X$ and $x \in X$. Notice that C is a fuzzy consequence operator in Ω' if and only if $\varphi(C)$ is a fuzzy consequence operator in Ω'^{δ}. The same is true for fuzzy interior operators, C is a fuzzy interior operator in Ω' if and only if $\varphi(C)$ is an fuzzy interior operator in Ω'^{δ}. Therefore, every result stated for fuzzy consequence operators in Ω' have its dual statement, true for fuzzy interior operators in Ω'^{δ} which becomes also true for fuzzy interior operators in Ω' via φ^{-1}. In fuzzy mathematical morphology, both kinds of operators act as morphological filters for image processing [15,16]. They have been extensively studied in several contexts [6,26,33] and they have been used to transfer results from the field of approximate reasoning to the field of fuzzy mathematical morphology [20].

Let us recall the definition of the fuzzy closure of a fuzzy operator. This notion was first defined for general lattices [35] and later translated to the fuzzy context by Pavelka [30]. It can be thought as the best upper approximation by a fuzzy consequence operator to a given operator.

Definition 2.3. Let $C : [0,1]^X \longrightarrow [0,1]^X$ be a fuzzy operator. We define the fuzzy closure $\overline{C}$ of C as the fuzzy operator given by

$$\overline{C} = \inf_{\substack{\phi \in \Omega \\ C \leqslant \phi}} \{\phi\}. \tag{1}$$

The fuzzy closure is a fuzzy consequence operator and it is uniquely determined since the infimum of fuzzy consequence operators so is. Dually, one can consider the greatest fuzzy interior operator which is smaller than or equal to a given operator; that is the best lower approximation of a fuzzy operator C by a fuzzy interior operator.

Definition 2.4. Let $C : [0,1]^X \longrightarrow [0,1]^X$ be a fuzzy operator. We define the fuzzy interior $\underline{C}$ of C as the fuzzy operator given by

$$\underline{C} = \sup_{\substack{\phi \in \Lambda \\ C \geqslant \phi}} \{\phi\}. \tag{2}$$

Fuzzy (binary) relations on X are fuzzy subsets of the cartesian product $X \times X$. For every pair $(x,y) \in X \times X$, $R(x,y)$ represents the degree in which x is related to y. We denote Γ' the set of fuzzy binary relations defined on X.

Definition 2.5. A fuzzy relation $R : X \times X \longrightarrow [0,1]$ is called a fuzzy $*$-preorder if it satisfies:

- Reflexivity: $R(x,x) = 1 \ \forall x \in X$.
- $*$-Transitivity: $R(x,y) * R(y,z) \leqslant R(x,z) \ \forall x,y,z \in X$.

A fuzzy preorder is called a fuzzy $*$-indistinguishability relation or fuzzy $*$-similarity if it also satisfies.

- Symmetry: $R(x,y) = R(y,x)\ \forall x,y \in X$.

Recall that for R and $S \in \Gamma'$, we say that $R \leqslant S$ if $R(x,y) \leqslant S(x,y)$ for all $x,y \in X$.

We will consider in Γ' the sup-$*$ composition which was introduced by Zadeh [36].

Definition 2.6. Let $R,S \in \Gamma'$ be fuzzy relations on a set X and $*$ a t-norm. The sup-$*$ composition of R and S is the fuzzy relation defined for all $x,y \in X$ by

$$R \circ S(x,y) = \sup_{w \in X}\{R(x,w) * S(w,y)\} \tag{3}$$

For a given fuzzy relation R, a fuzzy subset μ of X is called $*$-compatible with R if $\mu(x) * R(x,y) \leqslant \mu(y)$ for all $x,y \in X$. From its logical implications, these sets are also called true-sets or closed under modus ponens. This notion gets special interest when R is a preorder [11]. When R is not only a preorder but also an indistinguishability relation, these sets are called *extensional sets* and they have been largely studied [31].

3. Connections between fuzzy relations and fuzzy operators

Concepts of fuzzy relations and fuzzy operators are closely related. Zadeh as early as 1973 [37] introduced the Compositional Rule of Inference (CRI) that generates a fuzzy operator from a given fuzzy relation. Since then, the study of the relation between fuzzy relations and fuzzy operators has been a very fruitful area of research and applications. If we restrict to the concepts of Consequence and Interior operators, relevant results have been obtained for fuzzy $*$-similarities in [4,12,27,31], for fuzzy $*$-preorders in [8,18,19], and for general fuzzy relations in [7]. We shall focus on the operators C_R^* and $C_R^{\rightarrow}$ induced by a relation R.

3.1. The operator C_R^*

Every fuzzy relation induces a fuzzy operator through the well-known Zadeh's rule of inference [38].

Definition 3.1. Let $R \in \Gamma'$ be a fuzzy relation on X. The fuzzy operator induced by R through Zadeh's compositional rule is defined by

$$C_R^*(\mu)(x) = \sup_{w \in X}\{\mu(w) * R(w,x)\} \tag{4}$$

Notice that from a logical point of view, C_R^* can be understood as the operator that sends every fuzzy set μ to the fuzzy set containing all the elements which are related to some element w in μ by means of the relation R.

Proposition 3.1 [19]. *Let $\sigma : \Gamma' \longrightarrow \Omega'$ be the function that sends every fuzzy relation R to the operator C_R^* induced by means of Eq. (4). Then, σ is injective.*

In other words, injectivity of σ states that for any two fuzzy relations R and S, we have $C_R^* = C_S^*$ if and only if $R = S$. The relationship between fuzzy preorders and fuzzy consequence operators was well established [18,11].

Proposition 3.2. *Let R be a fuzzy relation. Then C_R^* is a fuzzy consequence operator if and only if R is a fuzzy $*$-preorder.*

It is worth recalling that not all FCO can be obtained from fuzzy preorders by means of Zadeh's compositional rule. When the starting relation is a fuzzy indistinguishability relation, the induced operator is not only a FCO but satisfies the following properties [31].

Proposition 3.3. *Let E be a fuzzy $*$-indistinguishability relation and let C_E^* be the fuzzy operator induced through Zadeh's compositional rule. Then,*

1. *C_E^* is a fuzzy consequence operator.*
2. *$C_E^*(\bigvee_{i \in I}\mu_i) = \bigvee_{i \in I} C_E^*(\mu_i)$ for any index set I and all $\mu_i \in [0,1]^X$.*
3. *$C_E^*(\{x\})(y) = C_E^*(\{y\})(x)$ for all $x,y \in X$ where $\{x\}$ denotes the singleton of x.*
4. *$C_E^*(\alpha * \mu) = \alpha * C_E^*(\mu)$ for any constant $\alpha \in [0,1]$ and $\mu \in [0,1]^X$.*

Proposition 3.4. *There is a bijection between the set of $*$-indistinguishability relations and the set of fuzzy operators satisfying the conditions of Proposition 3.3.*

We generalize the operator induced by a fuzzy relation through Zadeh's compositional rule to a fuzzy operator induced by a fuzzy relation and another fuzzy operator.

Definition 3.2. Let $g \in \Omega'$ be a fuzzy operator and let $R \in \Gamma'$ be a fuzzy relation on X. We define the operator C_R^g induced by g and R as

$$C_R^g(\mu)(x) = \sup_{w \in X}\{g(\mu)(w) * R(w,x)\} \tag{5}$$

R and g are called the generators of C_R^g.

The operator g used as generator performs a selection in order to apply Zadeh's usual operator only to the fuzzy subsets of its image. Notice that taking $g = id$, where id denotes the identity operator on $[0,1]^X$, we obtain $C_R^{id} = C_R^*$.

Proposition 3.5. *For every $g \in \Omega'$, the mapping $\sigma_g : \Gamma' \longrightarrow \Omega'$ that sends every fuzzy relation R to the operator C_R^g induced by R and g by means of Eq. (5) is increasing. That is, if $R \leqslant S$ then $C_R^g \leqslant C_S^g$.*

Proof. It directly follows from the monotonicity of $*$. □

Corollary 3.1. *The mapping $\sigma : \Gamma' \longrightarrow \Omega'$ that sends every fuzzy relation R to the operator C_R^* induced by Zadeh's compositional rule (Eq. (4)) is increasing.*

Our interest lies in the obtention of fuzzy consequence operators. For this, we need certain individual properties of the generators and also some conditions involving both generators, operators and relations. More precisely, let us define the concordance between a fuzzy operator and a fuzzy relation.

Definition 3.3. Let g be a fuzzy operator and R a fuzzy relation. We will say that g is $*$-*concordant with* R if all the subsets from the image of g are $*$-compatible with R. That is,

$$g(\mu)(x) * R(x,y) \leqslant g(\mu)(y)$$

for all $x, y \in X$ and all $\mu \in [0,1]^X$.

Theorem 3.1. *Let $R \in \Gamma'$ be a reflexive fuzzy relation and let $g \in \Omega'$ be a FCO. Suppose that g is $*$-concordant with R. Then, the operator C_R^g induced by g and R is also a FCO.*

Proof. Let us start proving the inclusion and monotonicity properties. From the reflexivity of R, it follows that

$$C_R^g(\mu)(x) = \sup_{w \in X}\{g(\mu)(w) * R(w,x)\} \geqslant g(\mu)(x) * R(x,x) = g(\mu)(x).$$

Since g is a FCO and therefore inclusive, we get

$$C_R^g(\mu)(x) \geqslant g(\mu)(x) \geqslant \mu(x)$$

Let $\mu_1, \mu_2 \in [0,1]^X$ such that $\mu_1 \subseteq \mu_2$. From the monotonicity of g it follows that $g(\mu_1)(x) \leqslant g(\mu_2)(x)$ for all $x \in X$. Therefore,

$$C_R^g(\mu_1)(x) = \sup_{w \in X}\{g(\mu_1)(w) * R(w,x)\} \leqslant \sup_{w \in X}\{g(\mu_2)(w) * R(w,x)\} = C_R^g(\mu_2)(x).$$

It only remains to prove the idempotence. To prove the first inclusion notice that, since $g(\mu)$ belongs to $\mathrm{Im}(g)$, it is $*$-compatible with R. That is,

$$g(\mu)(y) * R(y,x) \leqslant g(\mu)(x)$$

for all $y, x \in X$. Hence,

$$\sup_{y \in X}\{g(\mu)(y) * R(y,x)\} \leqslant g(\mu)(x)$$

for all $x \in X$. Using this fact, the monotonicity and idempotence of g and the monotonicity of $*$ we get

$$\begin{aligned} C_R^g(C_R^g(\mu))(x) &= \sup_{w \in X}\{g(C_R^g(\mu))(w) * R(w,x)\} = \sup_{w \in X}\left\{ g\left(\sup_{y \in X}\{g(\mu)(y) * R(y,w)\}\right) * R(w,x)\right\} \leqslant \sup_{w \in X}\{g(g(\mu)(w)) * R(w,x)\} \\ &= \sup_{w \in X}\{g(\mu)(w) * R(w,x)\} = C_R^g(\mu)(x) \end{aligned}$$

The other inclusion follows immediately from the inclusion property. □

3.2. The operator $C_R^{\rightarrow}$

Instead of using the supremum and the t-norm, one can induce a fuzzy operator from a fuzzy relation using the infimum and the adjoined implication.

Definition 3.4. Let $R \in \Gamma'$ be a fuzzy relation on X. We define the fuzzy operator induced by R through the inf-$\rightarrow$ composition as

$$C_R^{\rightarrow}(\mu)(x) = \inf_{w \in X}\{R(x,w) \rightarrow \mu(w)\} \tag{6}$$

Given a fuzzy set $\mu, C_R^{\rightarrow}(\mu)$ is the fuzzy subset containing the elements x such that whenever x is in relation through R with an element w, then w belongs to μ [17].

Proposition 3.6. *The mapping $\theta : \Gamma' \longrightarrow \Omega'$ that sends every fuzzy relation R to the operator $C_R^{\rightarrow}$ induced by means of Eq. (6) is decreasing. That is, if $R \leqslant S$ then $C_R^{\rightarrow} \geqslant C_S^{\rightarrow}$.*

Proof. It follows from the fact that $\rightarrow$ is antitone in the first argument. □

Proposition 3.7. *The function $\theta : \Gamma' \longrightarrow \Omega'$ that sends every fuzzy relation R to the operator $C_R^{\rightarrow}$ induced by means of Eq. (6) is injective. That is, if $C_R^{\rightarrow} = C_S^{\rightarrow}$ then $R = S$.*

Proof. We shall prove the contra-positive form that is, if $R \neq S$ necessarily $C_R^{\rightarrow} \neq C_S^{\rightarrow}$. Assume $R \neq S$. Then, there exists $x, y \in X$ such that $R(x,y) \neq S(x,y)$. We can suppose without loss of generality that $R(x,y) > S(x,y)$. Let us define the fuzzy set μ_x as $\mu_x(w) = S(x,w)$. Then,

$$C_S^{\rightarrow}(\mu_x)(x) = \inf_{w \in X}\{S(x,w) \rightarrow \mu_x(w)\} = \inf_{w \in X}\{S(x,w) \rightarrow S(x,w)\} = 1$$

but

$$C_R^{\rightarrow}(\mu_x)(x) = \inf_{w \in X}\{R(x,w) \rightarrow \mu_x(w)\} = \inf_{w \in X}\{R(x,w) \rightarrow S(x,w)\} \leqslant R(x,y) \rightarrow S(x,y) < 1$$

by Property 2 from Proposition 2.1. □

Again, fuzzy operators induced by fuzzy preorders or fuzzy indistinguishabilities satisfy certain special properties. They will allow us to connect permutability of fuzzy relations with permutability of fuzzy operators. It is known that the operator $C_R^{\rightarrow}$ is a FIO whenever R is a $*$-indistinguishability relation. The following result shows that it is enough that R is a fuzzy preorder. Though the result is implicitly stated in Proposition 15 of [8] and has been proved for fuzzy $*$-indistinguishability relations in Corollary 5.8 of [9], we present an alternative proof of it.

Proposition 3.8. *Let R be a preorder, then $C_R^{\rightarrow}$ defined as in (6) is a fuzzy interior operator.*

Proof. Let us first proof anti-inclusion and monotonicity.

$$C_R^{\rightarrow}(\mu)(x) = \inf_{w \in X}\{R(x,w) \rightarrow \mu(w)\} \leqslant R(x,x) \rightarrow \mu(x) \leqslant 1 \rightarrow \mu(x) = \mu(x)$$

Let $\mu, \nu \in [0,1]^X$ and assume $\mu \leqslant \nu$. Since $\rightarrow$ is monotone in the second argument, we have

$$R(x,w) \rightarrow \mu(w) \leqslant R(x,w) \rightarrow \nu(w) \quad \forall w \in X.$$

Therefore

$$C_R^{\rightarrow}(\mu)(x) = \inf_{w \in X}\{R(x,w) \rightarrow \mu(w)\} \leqslant \inf_{w \in X}\{R(x,w) \rightarrow \nu(w)\} = C_R^{\rightarrow}(\nu)(x)$$

To prove idempotence notice that

$$\begin{aligned} C_R^{\rightarrow}(C_R^{\rightarrow}(\mu))(x) &= \inf_{w \in X}\{R(x,w) \rightarrow C_R^{\rightarrow}(\mu)(w)\} = \inf_{w \in X}\left\{R(x,w) \rightarrow \left(\inf_{y \in X}\{R(w,y) \rightarrow \mu(y)\}\right)\right\} = \inf_{w \in X}\inf_{y \in X}\{R(x,w) \rightarrow (R(w,y) \\ &\rightarrow \mu(y))\} = \inf_{w \in X}\inf_{y \in X}\{(R(x,w) * R(w,y)) \rightarrow \mu(y)\} \geqslant \inf_{w \in X}\inf_{y \in X}\{R(x,y) \rightarrow \mu(y)\} = \inf_{y \in X}\{R(x,y) \rightarrow \mu(y)\} \\ &= C_R^{\rightarrow}(\mu)(x). \end{aligned}$$

The other inclusion follows from the anti-inclusion property. □

Proposition 3.9 [31]. *Let $E \in \Gamma'$ be a fuzzy $*$-indistinguishability relation and let $C_E^{\rightarrow}$ be the fuzzy operator induced by means of Eq. (6). Then $C_E^{\rightarrow}(\mu)(x)$ satisfies the following properties:*

1. *$C_E^{\rightarrow}$ is a fuzzy interior operator.*
2. *$C_E^{\rightarrow}(\bigwedge_{i\in I}\mu_i) = \bigwedge_{i\in I}C_E^{\rightarrow}(\mu_i)$ for any index set I and all $\mu_i \in [0,1]^X$.*
3. *$C_E^{\rightarrow}(\{x\} \rightarrow \alpha)(y) = C_E^{\rightarrow}(\{y\} \rightarrow \alpha)(x)$ for all $x, y \in X$ and any constant $\alpha \in [0,1]$ where $\{x\}$ denotes the singleton of x.*
4. *$C_E^{\rightarrow}(\alpha \rightarrow \mu) = \alpha \rightarrow C_E^{\rightarrow}(\mu)$ for any constant $\alpha \in [0,1]$ and $\mu \in [0,1]^X$.*

The converse of Proposition 3.9 also holds.

Proposition 3.10 [31]. *There exists a bijection between the set of fuzzy $*$-indistinguishability relations and the set of fuzzy operators satisfying all the properties from Proposition 3.9. That is, if $C \in \Omega'$ is a fuzzy operator satisfying all the properties from Proposition 3.9, then there exists a fuzzy $*$-indistinguishability relation E such that $C = C_E^{\rightarrow}$.*

4. Permutability of fuzzy preoders and fuzzy indistinguishability relations

In this section we study the permutability of fuzzy preoders and fuzzy indistinguishability relations. This will allow us a further analysis about the permutability of the operators C_R^* and $C_R^{\rightarrow}$.

Definition 4.1. Let $R, S \in \Gamma'$ be fuzzy relations. We say that R and S are permutable or that R and S permute if $R \circ S = S \circ R$ where $\circ$ is the sup-$*$ composition as in Eq. (3).

Permutability of preorders is closely related to the transitive closure of a fuzzy relation. The transitive closure of a fuzzy relation R is the smallest upper approximation of R which is $*$-transitive [2]. More precisely,

Definition 4.2. Let R be a fuzzy relation. We define the transitive closure $\overline{R}$ of R as the fuzzy relation given by

$$\overline{R} = \inf_{\substack{S\in\hat{\Gamma} \\ R\leqslant S}} \{S\} \tag{7}$$

where $\widehat{\Gamma}$ denotes the set of all $*$-transitive fuzzy relations on X.

The explicit formula for the transitive closure is given by $\overline{R} = \sup_{n\in\mathbb{N}} R^n$ where the power of R is defined using the sup-$*$ composition [2]. It is the smallest transitive relation greater than or equal to R. The $*$-transitive closure preserves reflexivity and symmetry. Hence, the transitive closure of a reflexive fuzzy relation is fuzzy preorder and the transitive closure of a reflexive and symmetric relation is an indistinguishability relation.

It was proved in [32] that two fuzzy $*$-indistinguishability relations defined on a finite set X permute if and only if $E \circ F$ is an $*$-indistinguishability relation. In this case, $E \circ F = \overline{\max(E,F)}$. We extend this result to general fuzzy preorders and any set X, finite or not. Please note that the result is closely related to Theorem 19 of [8]. We need the following lemma.

Lemma 4.1. *Let R and P be two fuzzy $*$-preorders on a set X. Then, $R \circ P \leqslant \overline{\max(R,P)}$.*

Proof

$$R \circ P \leqslant \max(R,P) \circ \max(R,P) \leqslant \sup_{n\in\mathbb{N}}(\max(R,P))^n = \overline{\max(R,P)} \qquad \square$$

Theorem 4.1. *Let R and P be two fuzzy $*$-preorders on X. Then, R and P are permutable if and only if $R \circ P$ and $P \circ R$ are fuzzy $*$-preorders. Moreover, $R \circ P$ is a fuzzy $*$-preorder if and only if it coincides with the $*$-transitive closure $\overline{\max(R,P)}$ of $\max(R,P)$.*

Proof. Let us first prove the second statement. That is, $R \circ P$ is a fuzzy $*$-preorder if and only if it coincides with $\overline{\max(R,P)}$. Suppose that $R \circ P$ is a fuzzy $*$-preorder. Since $R \circ P \geqslant R$ and $R \circ P \geqslant P$ we have that $R \circ P \geqslant \max(R,P)$. As $R \circ P$ is a fuzzy preorder, it follows that $R \circ P \geqslant \overline{\max(R,P)}$. From Lemma 4.1, we get $R \circ P = \overline{\max(R,P)}$. The other implication follows from the fact that $\max(R,P)$ is reflexive and therefore $\overline{\max(R,P)}$ is a preorder.

Now, let us prove that R and P are permutable if and only if $R \circ P$ and $P \circ R$ are fuzzy $*$-preorders. Assume that $R \circ P = P \circ R$ and let us show that they are fuzzy preorders.

- Reflexivity:
$$R \circ P(x,x) = \sup_{w\in X}\{R(x,w) * P(w,x)\} \geqslant R(x,x) * P(x,x) = 1$$

- $*$-Transitivity: Since R is $*$-transitive, $\sup_{w\in X}\{R(x,w)*R(w,y)\}\leqslant R(x,y)$. The same holds for P. Thus,

$$\begin{aligned} R\circ P(x,y)*R\circ P(y,z) &= \sup_{w\in X}\{R(x,w)*P(w,y)\}*\sup_{h\in X}\{R(y,h)*P(h,z)\} = \sup_{w,h\in X}\{R(x,w)*P(w,y)*R(y,h)*P(h,z)\} \\ &\leqslant \sup_{w,h\in X}\{R(x,w)*(P\circ R)(w,h)*P(h,z)\} = \sup_{w,h\in X}\{R(x,w)*(R\circ P)(w,h)*P(h,z)\} \\ &= \sup_{w,h,y\in X}\{R(x,w)*R(w,y)*P(y,h)*P(h,z)\} \\ &= \sup_{y\in X}\left\{\sup_{w\in X}\{R(x,w)*R(w,y)\}*\sup_{h\in X}\{P(y,h)*P(h,z)\}\right\}\leqslant \sup_{y\in X}\{R(x,y)*P(y,z)\} = R\circ P(x,z). \end{aligned}$$

Hence, it follows that $R\circ P=\overline{\max(R,P)}=P\circ R$.

The other direction is straightforward. □

For fuzzy indistinguishability relations, the symmetric property facilitates the way. We need just to find that one of the compositions is an indistinguishability relation to get both of them.

Corollary 4.1. *Let E and F be two $*$-indistinguishability relations on X. Then, E and F are permutable if and only if $E\circ F$ is a $*$-indistinguishability relation. Moreover, this occurs if and only if $E\circ F$ coincides with the $*$-transitive closure $\overline{\max(E,F)}$ of $\max(E,F)$.*

Proof. Since E and F are fuzzy preorders, Theorem 4.1 ensures that they permute if and only if $E\circ F=\overline{\max(E,F)}=F\circ E$. Since $\max(E,F)$ is reflexive and symmetric, $\overline{\max(E,F)}$ is an indistinguishability relation. □

5. Permutability of fuzzy consequence operators and fuzzy interior operators

The aim of this section is to study when two fuzzy consequence operators or two fuzzy interior operators permute. Permutability of fuzzy operators is considered with the usual composition. That is,

Definition 5.1. Let C,C' be fuzzy operators. We say that C and C' are permutable or that C and C' permute if $C\circ C'=C'\circ C$ where $\circ$ denotes the usual composition.

In order to study permutability for these two cases we need to recall the definition of the power of a fuzzy operator and several of its properties.

Definition 5.2. Let $C:[0,1]^X\longrightarrow[0,1]^X$ be a fuzzy operator. We define C^k for $k\in\mathbb{N}$ as the fuzzy operator defined recursively as:

1. $C^1=C$ i.e. $C^1(\mu)(x)=C(\mu)(x)\ \forall\mu\in[0,1]^X$ and $\forall x\in X$.
2. $C^k=C(C^{k-1})$ i.e. $C^k(\mu)(x)=C(C^{k-1}(\mu))(x)\ \forall\mu\in[0,1]^X,\forall x\in X$ and $k\geqslant 2$.

That is, C^k is the usual composition of the operator C with itself k times.

The following lemma is straightforward. It allows us to define the limit operator for the sequence of powers of either an inclusive or anti-inclusive operator.

Lemma 5.1. *Let $C:[0,1]^X\longrightarrow[0,1]^X$ be a fuzzy operator. Then,*

1. *If C is inclusive, then C^k is inclusive for all $k\in\mathbb{N}$.*
2. *If C is anti-inclusive, then C^k is anti-inclusive for all $k\in\mathbb{N}$.*
3. *If C is monotone, then C^k is monotone for all $k\in\mathbb{N}$.*

Proposition 5.1. *Let $C:[0,1]^X\longrightarrow[0,1]^X$ be a fuzzy operator.*

1. *If C is inclusive, then the sequence $\left\{C^k\right\}_{k\in\mathbb{N}}$ is increasing and convergent. That is, $C^k\leqslant C^{k+1}$ for all $k\in\mathbb{N}$ and there exists a fuzzy operator $U\in\Omega'$ such that $U=\lim_{n\in\mathbb{N}}C^n=\sup_{n\in\mathbb{N}}C^n$.*

2. *If C is anti-inclusive, then the sequence* $\left\{C^k\right\}_{k\in\mathbb{N}}$ *is decreasing and convergent. That is,* $C^{k+1} \leqslant C^k$ *for all* $k \in \mathbb{N}$ *and there exists a fuzzy operator* $L \in \Omega'$ *such that* $L = \lim_{n\in\mathbb{N}} C^n = \inf_{n\in\mathbb{N}} C^n$.

Proof.

1. Since 1 is an upper bound for $C^k(\mu)(x)$ for all $\mu \in [0,1]^X$, all $x \in X$ and all $k \in \mathbb{N}$, the sequences $\left\{C^k(\mu)(x)\right\}_{k\in\mathbb{N}}$ are increasing and bounded, thus they converge. Hence, the limit operator exists and it is pointwise defined by

$$U(\mu)(x) = \lim_{n\to\infty} C^n(\mu)(x) = \sup_{n\in\mathbb{N}} C^n(\mu)(x). \tag{8}$$

2. Dual to the previous one. In this case, 0 is a lower bound for $C^k(\mu)(x)$ for all $\mu \in [0,1]^X$, all $x \in X$ and all $k \in \mathbb{N}$. Therefore, the limit operator exists and it is pointwise defined by

$$L(\mu)(x) = \lim_{n\to\infty} C^n(\mu)(x) = \inf_{n\in\mathbb{N}} C^n(\mu)(x). \qquad \Box \tag{9}$$

5.1. Permutability of fuzzy consequence operators

Now we are ready to characterize permutability for fuzzy consequence operators. We shall see that the closure of an operator plays an essential role for permutability.

Similarly to the transitive closure of a fuzzy relation, the closure of certain operators can be defined from its sequence of powers.

Theorem 5.1. *Let* $C : [0,1]^X \longrightarrow [0,1]^X$ *be an inclusive and monotone fuzzy operator. Then,* $\lim_{n\to\infty} C^n = \overline{C}$.

Proof. First of all, let us show that $C^k \leqslant \overline{C}$ for all $k \in \mathbb{N}$ by induction on k.

- For $k = 1$ it is clear that $C \leqslant \overline{C}$.
- Assume that $C^k \leqslant \overline{C}$ for a certain k. Then, $C^k(\mu) \subseteq \overline{C}(\mu)$ for all $\mu \in [0,1]^X$. Since $C \leqslant \overline{C}$ and $\overline{C}$ is monotone and idempotent, it follows that

$$C(C^k(\mu)) \subseteq \overline{C}(C^k(\mu)) \subseteq \overline{C}(\overline{C}(\mu)) = \overline{C}(\mu).$$

Since $C^n \leqslant \overline{C}$ for all $n \in \mathbb{N}$, it follows that $\lim_{n\in\mathbb{N}} C^n \leqslant \overline{C}$.

To prove that $\lim_{n\to\infty} C^n \geqslant \overline{C}$ let us show that $\lim_{n\to\infty} C^n$ is a closure operator. Since C is inclusive and monotone, Lemma 5.1 ensures the inclusion and monotonicity of $\lim_{n\to\infty} C^n$. For the idempotence, it is straightforward that

$$\lim_{n\to\infty} C^n\left(\lim_{n\to\infty} C^n(\mu)\right)(x) = \lim_{n\to\infty} C^n(\mu)(x).$$

Therefore, $\lim_{n\to\infty} C^n = \sup_{n\in\mathbb{N}} C^n = \overline{C}$. $\Box$

Lemma 5.2. *Let* $C, C' : [0,1]^X \longrightarrow [0,1]^X$ *be fuzzy consequence operators. Then,*

$$C \circ C' \geqslant \max(C, C').$$

Proof. It directly follows from the inclusion and monotonicity properties. Since C is inclusive $C'(\mu) \subseteq C(C'(\mu))$ for all $\mu \in [0,1]^X$ and $C \circ C' \geqslant C'$. Since C' is inclusive $\mu \subseteq C'(\mu)$ and adding the monotonicity of C we get that $C(\mu) \subseteq C(C'(\mu))$ for all $\mu \in [0,1]^X$ and $C \circ C' \geqslant C$. Therefore, $C \circ C' \geqslant \max(C, C')$. $\Box$

Lemma 5.3. *Let* $C, C' : [0,1]^X \longrightarrow [0,1]^X$ *be two fuzzy consequence operators. Then,* $\max(C, C')$ *is an inclusive and monotone fuzzy operator.*

Proof. The proof is straightforward. As C and C' are inclusive, $\max(C, C')$ is also inclusive. For the monotonicity, note that $\mu_1 \subseteq \mu_2$ implies $C(\mu_1)(x) \leqslant C(\mu_2)(x)$ and $C'(\mu_1)(x) \leqslant C'(\mu_2)(x)$ for all $\mu \in [0,1]^X$ and $x \in X$. Hence, $\max(C, C')(\mu_1)(x)) \leqslant \max(C, C')(\mu_2)(x))$ for all $\mu \in [0,1]^X$ and $x \in X$. $\Box$

Remark 5.1. Notice that the two lemmas above hold even if C and C' are not FCO, but only inclusive and monotone. We did not use idempotence at any point of the proof.

The importance of the closure arises from the fact that preservation of the operator type through composition is the key for permuting two operators. We are ready to prove that there is only one case where the composition of two fuzzy consequence operators is again a fuzzy consequence operator.

Proposition 5.2. *Let C, C' be fuzzy consequence operators. Then, $C \circ C'$ is a fuzzy consequence operator if and only if $C \circ C' = \overline{\max(C, C')}$.*

Proof. It is sufficient to prove that if $C \circ C'$ is a FCO then $C \circ C' = \overline{\max(C, C')}$. The other implication follows from the fact that the closure of an operator is a FCO.

Assume that $C \circ C'$ is a FCO. From Lemma 5.2, $C \circ C' \geqslant \max(C, C')$. Therefore, $C \circ C' \geqslant \overline{\max(C, C')}$.

In addition, we have

$$C \circ C' \leqslant \max(C, C') \circ \max(C, C') = \max^2(C, C') \leqslant \overline{\max(C, C')}$$

where the last inequality holds due to Theorem 5.1 and Lemma 5.3. Hence, $C \circ C' = \overline{\max(C, C')}$. □

At this point, we are ready to characterize permutability of fuzzy consequence operators.

Theorem 5.2. *Let C, C' be fuzzy consequence operators. Then, C and C' permute if and only if $C \circ C'$ and $C' \circ C$ are fuzzy consequence operators.*

Proof. First, let us show that if C and C' permute, then $C \circ C'$ and $C' \circ C$ are FCO.

- Inclusion: From Lemmas 5.2 and 5.3, $C \circ C' \geqslant \max(C, C')$ which is inclusive.
- Monotonicity: Suppose $\mu_1 \subseteq \mu_2$. From the monotonicity of C' it follows that $C'(\mu_1) \subseteq C'(\mu_2)$ and from the monotonicity of $C, C(C'(\mu_1)) \subseteq C(C'(\mu_2))$.
- Idempotence:

$$(C \circ C')((C \circ C')(\mu))(x) = (C \circ C')((C' \circ C)(\mu))(x) = C(C'(C'(C(\mu))))(x) = C(C'(C(\mu)))(x) = C(C(C'(\mu)))(x) = C(C'(\mu))(x)$$
$$= (C \circ C')(\mu)(x).$$

The same arguments hold for $C' \circ C$.

The other implication directly follows from Proposition 5.2. □

Remark 5.2. There are cases of fuzzy consequence operators C and C' such that $C' \circ C$ is a FCO (and therefore $C' \circ C = \overline{\max(C, C')}$) but C and C' do not permute. We illustrate this remark with the following example.

Example 5.1. Let X be a non empty classical set and let $\alpha, \beta \in \mathbb{R}$ such that $0 < \beta < \alpha < 1$.

Let C' and C be FCO defined as follows:

$$C'(\mu)(x) = \begin{cases} 1 & \text{if } \mu(x) > \beta \\ \beta & \text{if } \mu(x) \leqslant \beta \end{cases} \quad C(\mu)(x) = \begin{cases} 1 & \text{if } \mu(x) > \alpha \\ \alpha & \text{if } \mu(x) \leqslant \alpha \end{cases}$$

Notice that $C' \circ C = \overline{\max(C, C')} = X$ where $X(x) = 1$ for all $x \in X$, but $C' \circ C \neq C \circ C'$. In fact,

$$(C \circ C')(\mu)(x) \begin{cases} 1 & \text{if } \mu(x) > \beta \\ \alpha & \text{if } \mu(x) \leqslant \beta \end{cases}$$

which is not a FCO.

5.2. Permutability of fuzzy interior operators

Dual results can be obtained for fuzzy interior operators. In this case, preservation of the type of operator is related to the interior of the minimum.

Theorem 5.3. *Let $C : [0,1]^X \longrightarrow [0,1]^X$ be an anti-inclusive and monotone fuzzy operator. Then, $\lim_{n\to\infty} C^n = \underline{C}$.*

Proof. The proof is dual to Theorem 5.1, therefore we will only give a sketch of it. By induction on k, it can be proved that $C^k \geqslant \underline{C}$ for all $k \in \mathbb{N}$. Thus, $\lim_{n\to\infty} C^n \geqslant \underline{C}$.

To prove the other inequality we need to show that $\lim_{n\to\infty} C^n$ is an interior operator. Lemmas 2 and 3 ensure the anti-inclusion and monotonicity properties. The idempotence is obtained using the definition of limit as done in Theorem 5.1.

Hence, $\lim_{n\to\infty} C^n = \inf_{n\in\mathbb{N}} C^n = \underline{C}$. □

Lemma 5.4. *Let* $C, C' : [0,1]^X \longrightarrow [0,1]^X$ *be fuzzy interior operators. Then,*

$$C \circ C' \leqslant \min(C, C').$$

Lemma 5.5. *Let* $C, C' : [0,1]^X \longrightarrow [0,1]^X$ *be fuzzy interior operators. Then,* $\min(C, C')$ *is an anti-inclusive and monotone fuzzy operator.*

Again, permutability is connected to the preservation of the type of operator through composition. There is only one case for which the composition of two fuzzy interior operators is again a fuzzy interior operator. This determines when permutability appears.

Proposition 5.3. *Let* C, C' *be fuzzy interior operators. Then,* $C \circ C'$ *is a fuzzy interior operator if and only if* $C \circ C' = \underline{\min(C, C')}$.

Proof. The proof is analogous to Proposition 5.2. It is sufficient to prove that if $C \circ C'$ is a FIO then $C \circ C' = \underline{\min(C, C')}$. The other implication follows from the fact that the fuzzy interior of an operator is a FIO.

Suppose that $C \circ C'$ is a fuzzy interior operator. From Lemma 5.4, we know that $C \circ C' \leqslant \min(C, C')$. Therefore, $C \circ C' \leqslant \underline{\min(C, C')}$.

In addition, one has,

$$C \circ C' \geqslant \min(C, C') \circ \min(C, C') = \min^2(C, C') \geqslant \underline{\min(C, C')}$$

where the last inequality holds due to Theorem 5.3 and Lemma 5.5. Hence,

$$C \circ C' = \underline{\min(C, C')} \qquad \square$$

Theorem 5.4. *Let* C, C' *be fuzzy interior operators. Then,* C *and* C' *permute if and only if* $C \circ C'$ *and* $C' \circ C$ *are fuzzy interior operators.*

Proof. The proof is analogous to the proof of Theorem 5.2. First of all, let us show that if C and C' permute, then $C \circ C'$ and $C' \circ C$ are fuzzy interior operators. Monotonicity and idempotence are proved exactly in the same way than in Theorem 5.2. Inclusion follows from Lemmas 5.4 and 5.5. Since $C \circ C' \leqslant \min(C, C')$ and $\min(C, C')$ is anti-inclusive, so is $C \circ C'$. The same argument holds for $C' \circ C$.

The other implication directly follows from Proposition 5.3. □

6. Permutability of fuzzy operators induced by fuzzy relations through Zadeh's compositional rule

It is natural to think that permutability of fuzzy relations is connected to the permutability of their induced operators. We shall study these connections for the fuzzy operators C_R^* and $C_R^{\rightarrow}$ introduced in Section 3. Recall that for these cases, fuzzy consequence operators and fuzzy interior operators are obtained from fuzzy preorders and fuzzy indistinguishability relations. Let us start with the study of the operator induced through Zadeh's compositional rule.

The composition of two fuzzy operators induced through Zadeh's compositional rule can be expressed in terms of the sup-$*$ composition of the inducing relations.

Proposition 6.1. *Let* R, S *be two fuzzy relations and let* C_R^* *and* C_S^* *be the corresponding fuzzy operators induced through Zadeh's compositional rule. Then,*

$$C_R^* \circ C_S^* = C_{S\circ R}^* = C_R^{C_S^*} \qquad (10)$$

where $S \circ R$ *denotes the sup-$*$ product composition of fuzzy relations.*

Proof. For all $\mu \in [0,1]^X$ and all $x \in X$ we have

$$C_R^* \circ C_S^*(\mu)(x) = C_R^*(C_S^*(\mu))(x) = \sup_{w\in X} \{C_S^*(\mu)(w) * R(w,x)\} = C_R^{C_S^*}$$

which gives us the second equality. For the first one,

$$C_R^{C_S^*} = \sup_{w\in X} \{C_S^*(\mu)(w) * R(w,x)\} = \sup_{w\in X} \left\{ \sup_{z\in X}\{\mu(z) * S(z,w)\} * R(w,x) \right\} = \sup_{w,z\in X} \{\mu(z) * S(z,w) * R(w,x)\}$$
$$= \sup_{z\in X} \left\{ \mu(z) * \sup_{w\in X}\{S(z,w) * R(w,x)\} \right\} = \sup_{z\in X} \{\mu(z) * S\circ R(z,x)\} = C_{S\circ R}^*(\mu)(x). \qquad \square$$

The relation between permutability of fuzzy relations and permutability of their induced operators can be summarized in the following theorem.

Theorem 6.1. *Let R, S be two fuzzy relations and let C_R^* and C_S^* be the corresponding fuzzy operators induced through Zadeh's compositional rule. Then, C_R^* and C_S^* permute if and only if R and S permute.*

Proof. It follows directly from the fact that the function that sends each fuzzy relation R to its induced operator C_R^* is injective. Hence,

$$C_{S\circ R}^* = C_{R\circ S}^* \Longleftrightarrow S\circ R = R\circ S \qquad \square$$

As we have shown in the previous section, permutability of fuzzy consequence operators is related to the preservation of the type of operator. For fuzzy consequence operators induced by fuzzy preorders by means of Eq. (4) this occurs if and only if composition of the fuzzy preorders also preserves the type, i.e. it is again a fuzzy preorder.

Theorem 6.2. *Let R, P be fuzzy $*$-preorders and let C_R^* and C_P^* their corresponding fuzzy consequence operators induced through Zadeh's compositional rule. Then, C_R^* and C_P^* permute if and only if $R\circ P$ and $P\circ R$ are fuzzy $*$-preorders.*

Proof. From Theorem 6.1, $C_R^* \circ C_P^* = C_P^* \circ C_R^* \Longleftrightarrow R\circ P = P\circ R$ and from Theorem 4.1, $R\circ P = P\circ R$ if and only if both are fuzzy preorders. $\square$

Corollary 6.1. *Let R, P be fuzzy $*$-preorders and let C_R^* and C_P^* their corresponding fuzzy consequence operators induced through Zadeh's compositional rule. Then, C_R^* and C_P^* permute if and only if $R\circ P = P\circ R = \overline{\max(P,R)}$.*

The left implication of the previous corollary is a direct consequence of Theorem 19 in [8].

For permutability of fuzzy operators induced by fuzzy indistinguishability relations the following result holds.

Theorem 6.3. *Let E, F be fuzzy $*$-indistinguishability relations and let C_E^* and C_F^* be their corresponding fuzzy consequence operators induced through Zadeh's compositional rule. Then, C_E^* and C_F^* permute if and only if $E\circ F$ is a fuzzy $*$-indistinguishability relation.*

Proof. It directly follows from Corollary 4.1 and Theorem 6.2. $\square$

Corollary 6.2. *Let E, F be fuzzy $*$-indistinguishability relations and let C_E^* and C_F^* be their corresponding fuzzy consequence operators induced through Zadeh's compositional rule. Then, C_E^* and C_F^* permute if and only if $E\circ F = \overline{\max(E,F)}$.*

Corollary 6.3. *Let C, C' be fuzzy operators satisfying all the conditions of Proposition 3.3. Then, C and C' permute if and only if $C\circ C'$ also satisfies all these conditions.*

In Theorem 6.1, two different ways of writing the composition of fuzzy operators were presented. We shall see another approach to permutability that can be obtained using the second expression. This allows a sufficient condition for permutability in terms of the concordance between fuzzy relations and fuzzy operators, notion that we introduced in Definition 3.3.

Proposition 6.2. *Let R, P be fuzzy preorders and let C_R^* and C_P^* be their respective induced FCO by means of Eq. (4). If C_R^* is $*$-concordant with P and C_P^* is $*$-concordant with R, then P and R permute and therefore C_R^* and C_P^* also permute.*

Proof. It directly follows from Theorems 3.1 and 6.2. $\square$

The following theorem is adapted from [8]:

Theorem 6.4. *Let* $\{\mu_i\}_{i\in I} \subseteq [0,1]^X$ *be an arbitrary family of fuzzy subsets. Then,*

$$R(x,y) = \inf_{i\in I}\{\mu_i(x) \to \mu_i(y)\} \tag{11}$$

is the largest fuzzy preorder for which every fuzzy subset of the family $\{\mu_i\}_{i\in I}$ *is* $*$*-compatible with.*

Notice that $\{\mu_i\}_{i\in I}$ is also $*$-compatible with S for every fuzzy relation S smaller than or equal to (11). Using this result, we define the largest fuzzy preorder for which a given operator C can be $*$-concordant with.

Definition 6.1. Let C be a fuzzy operator in Ω'. The fuzzy relation R_c^c induced by C is given by

$$R_c^c(x,y) = \inf_{\mu\in[0,1]^X}\{C(\mu)(x) \to C(\mu)(y)\} \tag{12}$$

According to Theorem 6.4, the fuzzy preorder R_c^c defined above gives an upper bound which is sufficient for a relation to be $*$-concordant with the given operator C. Hence, if a fuzzy relation S is smaller than or equal to R_c^c for a certain fuzzy operator C, every fuzzy subset of the image of C will be compatible with S.

Proposition 6.3. *Let* S *be a fuzzy relation such that* $S \leqslant R_c^c$ *for a certain* $C \in \Omega'$*. Then,* C *is* $*$*-concordant with* S*.*

Proof. Straightforward. □

Corollary 6.4. *Let* R, P *be fuzzy preorders and let* C_R^* *and* C_P^* *be their respective induced FCO. If*

$$R \leqslant R_{C_P^*}^{C_P^*} \quad \text{and} \quad P \leqslant R_{C_R^*}^{C_R^*},$$

then R *and* P *permute. Therefore, so do* C_R^* *and* C_P^**.*

Proof. It directly follows from Propositions 6.2, 6.3 and Corollary 6.1.

7. Permutability of fuzzy operators induced by fuzzy relations through inf-$\to$ composition

Composition of operators induced by means of the inf-$\to$ composition as defined by (6) can be written in terms of the sup-$*$ composition of the inducing relations.

Proposition 7.1. *Let* R, S *be two fuzzy relations and let* $C_R^{\to}$ *and* $C_S^{\to}$ *be the corresponding fuzzy operators induced through the* inf - $\to$ *composition. Then,*

$$C_R^{\to} \circ C_S^{\to} = C_{R\circ S}^{\to} \tag{13}$$

where $S \circ R$ *denotes the sup-*$*$ *product composition of fuzzy relations.*

Proof. For all $\mu \in [0,1]^X$ and all $x \in X$ we have

$$\begin{aligned}
C_R^{\to} \circ C_S^{\to}(\mu)(x) &= \inf_{w\in X}\{R(x,w) \to C_S^{\to}(\mu)(x)\} = \inf_{w\in X}\left\{R(x,w) \to \left\{\inf_{y\in X}\{S(w,y) \to \mu(y)\}\right\}\right\} \\
&= \inf_{w,y\in X}\{R(x,w) \to \{S(w,y) \to \mu(y)\}\} = \inf_{y\in X}\inf_{w\in X}\{\{R(x,w) * S(w,y)\} \to \mu(y)\} \\
&= \inf_{y\in X}\left\{\sup_{w\in X}\{R(x,w) * S(w,y)\} \to \mu(y)\right\} = \inf_{y\in X}\{(R\circ S)(x,y) \to \mu(y)\} = C_{R\circ S}^{\to}(\mu)(x)
\end{aligned}$$

where most of the equalities follow from the properties in Proposition 2.1. □

As a consequence, we obtain similar results to the ones obtained for the operators induced by Zadeh's compositional rule.

Theorem 7.1. *Let* R, S *be two fuzzy relations and let* $C_R^{\to}$ *and* $C_S^{\to}$ *be the corresponding fuzzy operators induced through the* inf-$\to$ *composition. Then,* $C_R^{\to}$ *and* $C_S^{\to}$ *permute if and only if* R *and* S *permute.*

Proof. One the one side, assume that $S \circ R = R \circ S$. Then, from the previous proposition it follows that

$$C_R^{\rightarrow} \circ C_S^{\rightarrow}(\mu)(x) = C_{R\circ S}^{\rightarrow}(\mu)(x) = C_{S\circ R}^{\rightarrow}(\mu)(x) = C_S^{\rightarrow} \circ C_R^{\rightarrow}(\mu)(x)$$

On the other side, from Proposition 3.7, $C_{R\circ S}^{\rightarrow} = C_{S\circ R}^{\rightarrow}$implies $R \circ S = S \circ R$. □

Theorem 7.2. *Let R, P be fuzzy $*$-preorders and let $C_R^{\rightarrow}$ and $C_P^{\rightarrow}$ their corresponding fuzzy interior operators induced through the inf-$\longrightarrow$ composition by means of (6). Then, $C_R^{\rightarrow}$ and $C_P^{\rightarrow}$ permute if and only if $R \circ P$ and $P \circ R$ are fuzzy $*$-preorders.*

Proof. It directly follows from Theorems 4.1 and 7.1. □

Corollary 7.1. *Let R, P be fuzzy $*$-preorders and let $C_R^{\rightarrow}$ and $C_P^{\rightarrow}$ the corresponding fuzzy interior operators induced by means of (6). Then, $C_R^{\rightarrow}$ and $C_P^{\rightarrow}$ permute if and only if $R \circ P = P \circ R = \overline{\max(P, R)}$.*

The left implication of the previous corollary is a direct consequence of Theorem 19 in [8].

For permutability of fuzzy operators induced by fuzzy indistinguishability relations the following holds.

Theorem 7.3. *Let E, F be fuzzy $*$-indistinguishability relations and let $C_E^{\rightarrow}$ and $C_F^{\rightarrow}$ be their corresponding fuzzy interior operators induced by means of (6). Then, $C_E^{\rightarrow}$ and $C_F^{\rightarrow}$ permute if and only if $E \circ F$ is a fuzzy $*$-indistinguishability relation.*

Proof. It directly follows from Corollary 4.1 and Theorem 7.2. □

Corollary 7.2. *Let E, F be fuzzy $*$-indistinguishability relations and let $C_E^{\rightarrow}$ and $C_F^{\rightarrow}$ be their corresponding fuzzy interior operators induced by means of (6). Then, $C_E^{\rightarrow}$ and $C_F^{\rightarrow}$ permute if and only if $E \circ F = \overline{\max(E, F)}$.*

Corollary 7.3. *Let C, C' be fuzzy operators satisfying all the conditions of Proposition 3.9. Then, C and C' permute if and only if $C \circ C'$ also satisfies all these conditions.*

8. Composition without permutability

In the previous sections, we have seen that for both fuzzy consequence operators and fuzzy interior operators, permutability is completely connected with the preservation of the operator type through composition. That is, to obtain permutability of two fuzzy consequence operators, compositions in both directions must be fuzzy consequence operators again. Similarly, the compositions of two fuzzy interior operators must be fuzzy interior operators in order to find permutability between them. In the case of the fuzzy operators C_R^* and $C_R^{\rightarrow}$ induced by fuzzy relations, permutability of the fuzzy relations is a necessary and sufficient condition in order to find permutability between the induced operators.

Nevertheless, even when permutability does not appear certain properties are still transferred from the composition of relations to the composition of the induced operators.

Proposition 8.1. *Let R, P be fuzzy $*$-preorders. Then, the fuzzy operator $C_{P\circ R}^*$ is inclusive and monotone.*

Proof. From Lemmas 5.2 and 5.3, $C_{P\circ R}^* \geqslant \max(C_P^*, C_R^*)$ which is inclusive. Hence, so is $C_{P\circ R}^*$. To prove monotonicity assume $\mu \subseteq \nu$, then

$$C_{P\circ R}^*(\mu)(x) = \sup_{w\in X}\{\mu(w) * (P \circ R)(w, x)\} \leqslant \sup_{w\in X}\{\nu(w) * (P \circ R)(w, x)\} = C_{P\circ R}^*(\nu)(x) \qquad \square$$

Proposition 8.2. *Let E, F be fuzzy $*$-indistinguishability relations. Then, $C_{E\circ F}^*$ satisfies the inclusion and monotony properties from the definition of FCO. Moreover, it satisfies Properties 2, 4 of Proposition 3.3.*

Proof. Inclusion and monotonicity follows from Proposition 8.1. Since both C_E^* and C_F^* satisfy Properties 2 and 4, it follows that

$$C_E^*\left(C_F^*\left(\bigvee_{i\in I} \mu_i\right)\right) = C_E^*\left(\bigvee_{i\in I} C_F^*(\mu_i)\right) = \bigvee_{i\in I} C_E^*(C_F^*(\mu_i))$$

for any index set I and all $\mu_i \in [0, 1]^X$ and

$$C_E^*(C_F^*(\alpha * \mu)) = C_E^*(\alpha * C_F^*(\mu)) = \alpha * C_E^*(C_F^*(\mu))$$

for any constant $\alpha \in [0,1]$ and $\mu \in [0,1]^X$. □

We can weaken the conditions imposed to the inducing relations and study the composition of operators induced by reflexive fuzzy relations.

Proposition 8.3. *Let R, P be reflexive fuzzy relations. Then, the fuzzy operator $C^*_{P\circ R}$ is inclusive and monotone.*

Proof. Notice that proof of Proposition 8.1 holds for reflexive relations since we did not use $*$-transitivity. □

Remark 8.1. Observe that $C^*_{P\circ R}$is always monotone even if R and P are not reflexive.

This result allows us to connect the operator induced by the transitive closure of the sup-$*$ composition of two reflexive fuzzy relations with the closure of the operator that this composition induces in the following way.

Theorem 8.1. *Let R, P be reflexive fuzzy relations. Then, $\overline{C^*_{R\circ P}} = C^*_{\overline{R\circ P}}$.*

Proof. Since R and P are reflexive, so is $R\circ P$. Then, by the definition of closure, $R\circ P \leqslant \overline{R\circ P}$. Then, from Corollary 3.1 we get $C^*_{R\circ P} \leqslant C^*_{\overline{R\circ P}}$. As $\overline{R\circ P}$ is a fuzzy $*$-preoder, $C^*_{\overline{R\circ P}}$ is a FCO. Therefore, since the closure is the smallest FCO greater or equal to a given one we have $\overline{C^*_{R\circ P}} \leqslant C^*_{\overline{R\circ P}}$.

On the other side, from Proposition 8.3, $C^*_{R\circ P}$ is inclusive and monotone. Hence, by Proposition 5.1,

$$\overline{C^*_{R\circ P}} = \lim_{n\to\infty}(C^*_{R\circ P})^n = \sup_{n\in\mathbb{N}}(C^*_{R\circ P})^n \geqslant (C^*_{R\circ P})^n \quad \forall n \in \mathbb{N}.$$

From the recursive definition of the power of a fuzzy operator (Definition 5.2), we have $(C^*_{R\circ P})^n = C^*_{(R\circ P)^n}$, thus

$$\overline{C^*_{R\circ P}} = \sup_{n\in\mathbb{N}}(C^*_{R\circ P})^n \geqslant (C^*_{R\circ P})^n = C^*_{(R\circ P)^n} \quad \forall n \in \mathbb{N}.$$

Therefore, $\overline{C^*_{R\circ P}} \geqslant C^*_{\sup_{n\in\mathbb{N}}(R\circ P)^n} = C^*_{\overline{R\circ P}}$ and $\overline{C^*_{R\circ P}} = C^*_{\overline{R\circ P}}$. □

9. Conclusions

Permutability of fuzzy consequence operators and fuzzy interior operators does not always occur. However, there are cases for which the order of composition does not affect the result. We have shown that this fact is completely connected to the preservation of the operator type through composition.

For the particular cases of fuzzy consequence operators induced through Zadeh's compositional rule and fuzzy interior operators induced using the inf-$\rightarrow$ composition we proved that permutability of the relations is connected to permutability of the induced operators. In fact, permutability of the starting relations appears to be a necessary and sufficient condition in order to obtain permutability of the induced operators.

Finally, we have seen that for reflexive relations, the operator induced by the transitive closure of their composition coincides with the closure of the operator that their composition induces.

To conclude, we summarize the most important results that we have obtained. First of all, we enumerate the results about permutability of fuzzy preorders and fuzzy indistinguishability relations that have been the key to analyze permutability of their induced fuzzy operators:

1. R, P fuzzy $*$-preorders. Then, $R\circ P = P\circ R \iff R\circ P$ and $P\circ R$ are fuzzy $*$-preorders.
2. E, F fuzzy $*$-similarities. Then, $E\circ F = F\circ E \iff E\circ F$ is a fuzzy $*$-similarity.
3. $R\circ P$ preserves type (similarity or preorder) $\Leftrightarrow R\circ P = \overline{\max(R,P)}$.

Permutability of general fuzzy consequence and interior operators can be summarized in the following two results.

4. C, C' FCO. C and C' permute if and only if $C\circ C' = C'\circ C = \overline{\max(C,C')}$.
5. C, C' FIO. C and C' permute if and only if $C\circ C' = C'\circ C = \underline{\min(C,C')}$.
   Several results about permutability of fuzzy operators induced by fuzzy relations have been obtained. Results about permutability of fuzzy preorders and similarities allow some of the following characterizations. The notion of concordance between fuzzy relations and fuzzy operators also plays a relevant role (see item 11).
6. R, S fuzzy relations. $C^*_R \circ C^*_S = C^*_{S\circ R} = C_R^{C^*_S}$.

7. R, S fuzzy relations. $C_R^{\rightarrow} \circ C_S^{\rightarrow} = C_{R \circ S}^{\rightarrow}$.
8. R, S fuzzy relations. Then, $C_R^* \circ C_S^* = C_S^* \circ C_R^* \Longleftrightarrow R \circ S = S \circ R$.
9. R, S fuzzy relations. Then, $C_R^{\rightarrow} \circ C_S^{\rightarrow} = C_S^{\rightarrow} \circ C_R^{\rightarrow} \Longleftrightarrow R \circ S = S \circ R$.
10. R, S fuzzy $*$-preorders. Then, $C_R^* \circ C_S^* = C_S^* \circ C_R^* \Longleftrightarrow R \circ S$ and $S \circ R$ are fuzzy $*$-preorders.
11. R, P fuzzy $*$-preorders. C_R^* P $*$-concordant and C_P^* R $*$-concordant. Then, $C_R^* \circ C_P^* = C_P^* \circ C_R^*$.
12. E, F fuzzy $*$-indistinguishabilities. Then, $C_E^* \circ C_F^* = C_F^* \circ C_E^* \Longleftrightarrow E \circ F$ is a fuzzy $*$-indistinguishability.
13. R, S fuzzy $*$-preorders. Then, $C_R^{\rightarrow} \circ C_S^{\rightarrow} = C_S^{\rightarrow} \circ C_R^{\rightarrow} \Longleftrightarrow R \circ S$ and $S \circ R$ are fuzzy $*$-preorders.
14. E, F fuzzy $*$-indistinguishabilities. Then, $C_E^{\rightarrow} \circ C_F^{\rightarrow} = C_F^{\rightarrow} \circ C_E^{\rightarrow} \Longleftrightarrow E \circ F$ is a fuzzy $*$-indistinguishability.
15. R, S reflexive fuzzy relations. Then $\overline{C_{R \circ S}^*} = C_{\overline{S \circ R}}^*$.

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