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HYPERCHAOTIC SYNCHRONIZATION

GERARD VIDAL and HÉCTOR MANCINI
Departamento de Física y Matemática Aplicada,
Universidad de Navarra, Spain

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In this work we study the synchronization between identical pairs of hyperchaotic mathematical systems symmetrically coupled. Calculations are performed firstly in two well-known hyperchaotic systems, and then compared with the results obtained coupling symmetrically two Takens–Bogdanov systems (TBS) which represent a bifurcation in Codimension 2 (a point with two modes bifurcating simultaneously). In all of these systems, complete synchronization is achieved for some intervals of the coupling strength. As it will be shown, these windows can be localized by using the representation of the Lyapounov exponents against the coupling parameter. We analyze here these three models looking for general features in synchronization of hypercaotic systems, that could be useful to model mutual synchronization of two time-dependent convection experiments. We plan to use the results obtained in the TBS as a direct guide to control our experiment because this model was successfully used before to represent the observed dynamics. The other two systems presented here (Chen and Lü) are used to look for the possibility of general features and to check the used numerical methods.

Keywords: Hyperchaos; synchronization; complex systems.

1. Introduction

Hyperchaotic behavior was defined for a nonlinear system as a dynamical state for which more than one Lyapunov exponent becomes positive [Rössler, 1979]. This situation arises as a natural regime in extended space-time systems, delayed systems or in situations where many oscillators are coupled, a frequent situation in complex networks. In all these cases usually it is very difficult to understand what is happening physically inside the system for different reasons like the presence of symmetries restricting the possible solutions, or because delays are transforming the system into an infinite dimensional one. Sometimes, when the attractor presents some kind of symmetry properties, it is easier to adopt a more clear point of view about the system dynamics.

Synchronization between similar chaotic systems could make a collective state less complex than the state of the individual system for the same value of the control parameter. This result ranges from chaos suppression to different types of synchronization depending on whether the coupling is symmetric or not, and also from the strength of the coupling.

In past works we have shown that general synchronization can be achieved with chaos suppression through asymmetrical coupling between two identical low-dimensional chaotic systems for certain values of the coupling parameter [Bragard *et al.*, 2007]. In that work it was shown that it is possible to find these values by plotting the Lyapunov exponents against the coupling parameter.

More recently, we have studied the coupling dynamics in an hyperchaotic system with D_4 symmetry and we have shown that this four-dimensional system shows similar behavior becoming synchronized for some values of the coupling parameter

[Vidal & Mancini, 2009]. The work was performed on a mathematical model used in pattern formation processes [Hoyle, 2006] and it was applied in time-dependent patterns that appears in convection experiments where two modes lose stability simultaneously (codimension 2 TBS under square symmetry) [Ondarçuhu *et al.*, 1993]. The interplay between chaos and symmetry permits the use of some mathematical tools from group theory [Gilmore & Letellier, 2007], an approach very useful for discussing the results when the coupling factor strongly affects the system dynamics and modifies the attractor.

In the present work, we compare the effects of coupling two identical hyperchaotic oscillators of three different origins. Each oscillator is coupled symmetrically through the variable x. The original systems are two extended versions from the Lorenz model (Chen [Gao *et al.*, 2009] and Lü [Lü *et al.*, 2006]), a classical attractor used in convection, and the other one is the TBS mentioned before. All the systems are four-dimensional (x, y, z, w) with parameter values fixed to obtain an hyperchaotic behavior.

To perform numerical simulations we use a fourth order Runge–Kutta method with a $\Delta t = 10^{-2}$ in all cases. For calculating LE in all the systems we integrate the eight linearized equation sets of the coupled system. For these, we use the same Runge–Kutta method with the same Δt calculating the exponents in 10^8 time steps. The Gram–Schmidt normalization process is done every 50 time steps and this operation requires the most computational time. We reject a transitory state of 10^6 time steps before considering the steady state in the LE calculation. The number of time steps depends on the volume where our random initial condition is introduced. We have verified that this transitory is adequate for a good accuracy on calculations. Also, it is important to remark that some coupled systems become unstable for initial conditions that are outside even a sufficiently small region.

2. Chen System

In this section we discuss the Chen system [Zhou *et al.*, 2004] in one of its hyperchaotic versions. Concretely this work is based on the equation set appearing in [Gao *et al.*, 2009]. The difference between the original Chen and the hyperchaotic one is an extra variable (w). In the present work, the equation set is modified to correspond to a system with two coupled identical attractors. This could be mathematically written as follows:

$$\begin{aligned}
x'_{1,2} &= a(y_{1,2} - x_{1,2}) + \frac{\varepsilon}{2}(x_{2,1} - x_{1,2}) \\
y'_{1,2} &= -dx_{1,2} + x_{1,2}z_{1,2} + c(y_{1,2} - w_{1,2}) \\
z'_{1,2} &= x_{1,2}y_{1,2} - bz_{1,2} \\
w'_{1,2} &= x_{1,2} + k
\end{aligned} \tag{1}$$

As for the case studied above, the coupling is symmetrically done through x variables. The parameter values are $a = 36, b = 3, c = 28, d = 16$ and k being a fixed time delay in the extended variable. We have fixed the value of k for setting an hyperchaotic attractor [Gao *et al.*, 2009], there is another study of this attractor in [Gao *et al.*, 2008] where the parameter space is explored. For this attractor, we have calculated the LE and we have plotted these against the coupling factor in Fig. 1.

We have found, at least, four different dynamical behaviors depending on the coupling strength between both attractors. For distinguishing these cases, we focus our attention on the LE. The first case corresponds to the noncoupling system, the second one corresponds to the maximum of the LE, the third one corresponds to the coupling strength within the $[15, 30]$ interval and the fourth corresponds to values beyond $\varepsilon = 30$.

The first case is related to the free oscillator behavior and was studied in several papers as an introduction of the original system. Here we just

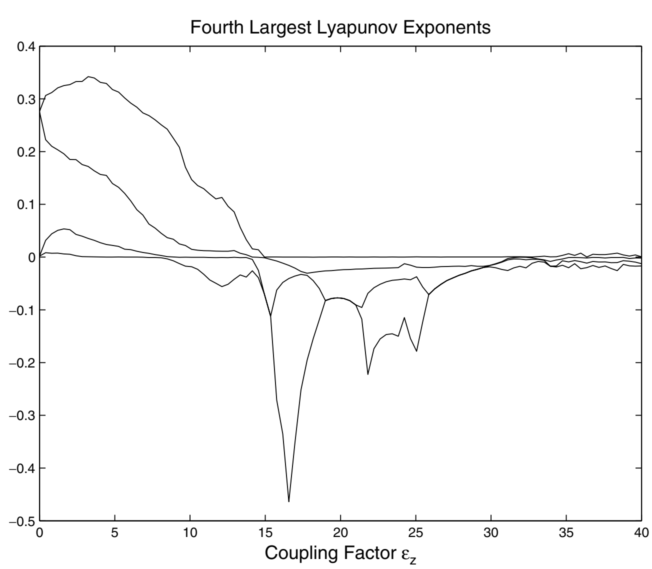
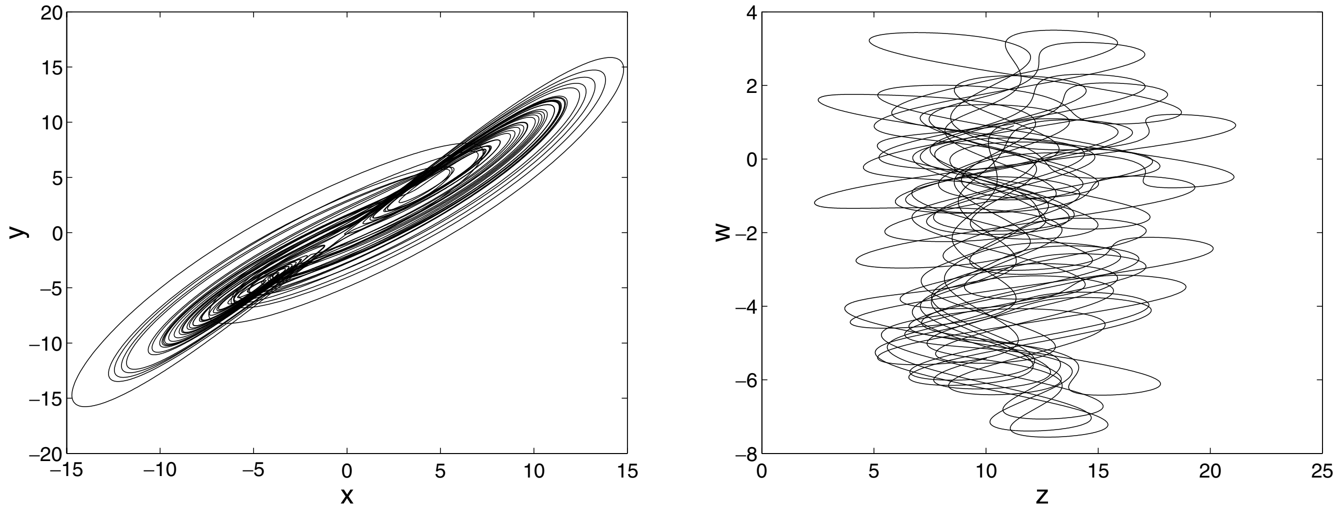


Fig. 1. Four largest LE against the coupling factor ε for the Chen hyperchaotic attractor.



(a) (b)

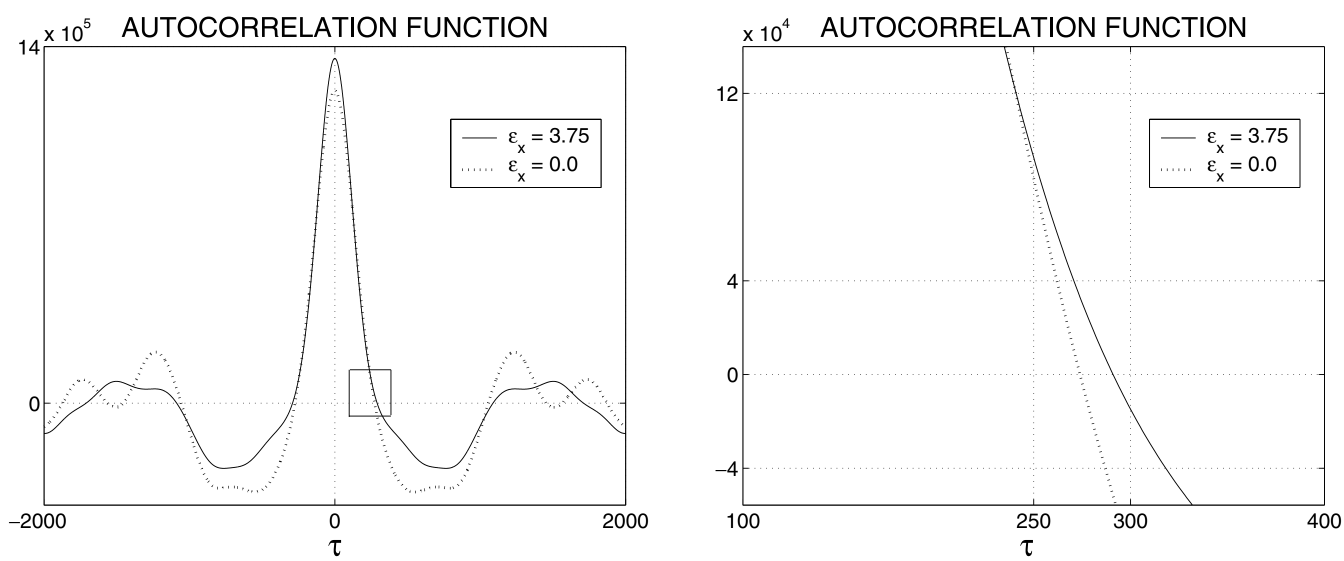
Fig. 2. This figure shows the (x, y) phase plane in (a) and the (z, w) in (b) for the uncoupled Chen's hyperchaotic system.

show in Fig. 2 a pair of phase planes for taking them as a reference and also for clarifying the changes in the dynamics by comparison with the others.

The second case is important in order to understand the meaning of the maximum in the LE against the coupling strength factor. Intuitively, larger LE implies more confused phase planes. But this is contradictory because the mutual information between both systems (S_1 and S_2) is given by the equation:

$$I(S_1; S_2) = H(S_1) + H(S_2) - H(S_2|S_1) \qquad (2)$$

$H(S_1)$ and $H(S_2)$ being the entropy of each system. So the most disordered scenario takes place when the systems are uncoupled and the mutual information is minimum. Although there are no significant differences among the phases planes, it



(a) (b)

Fig. 3. Autocorrelation of x signals in Chen's hyperchaotic system for a ε of 0 (dotted line) and 3.75 (solid line). There is an inset in (b) which shows that the first zero crossing occurs for the uncoupled system.

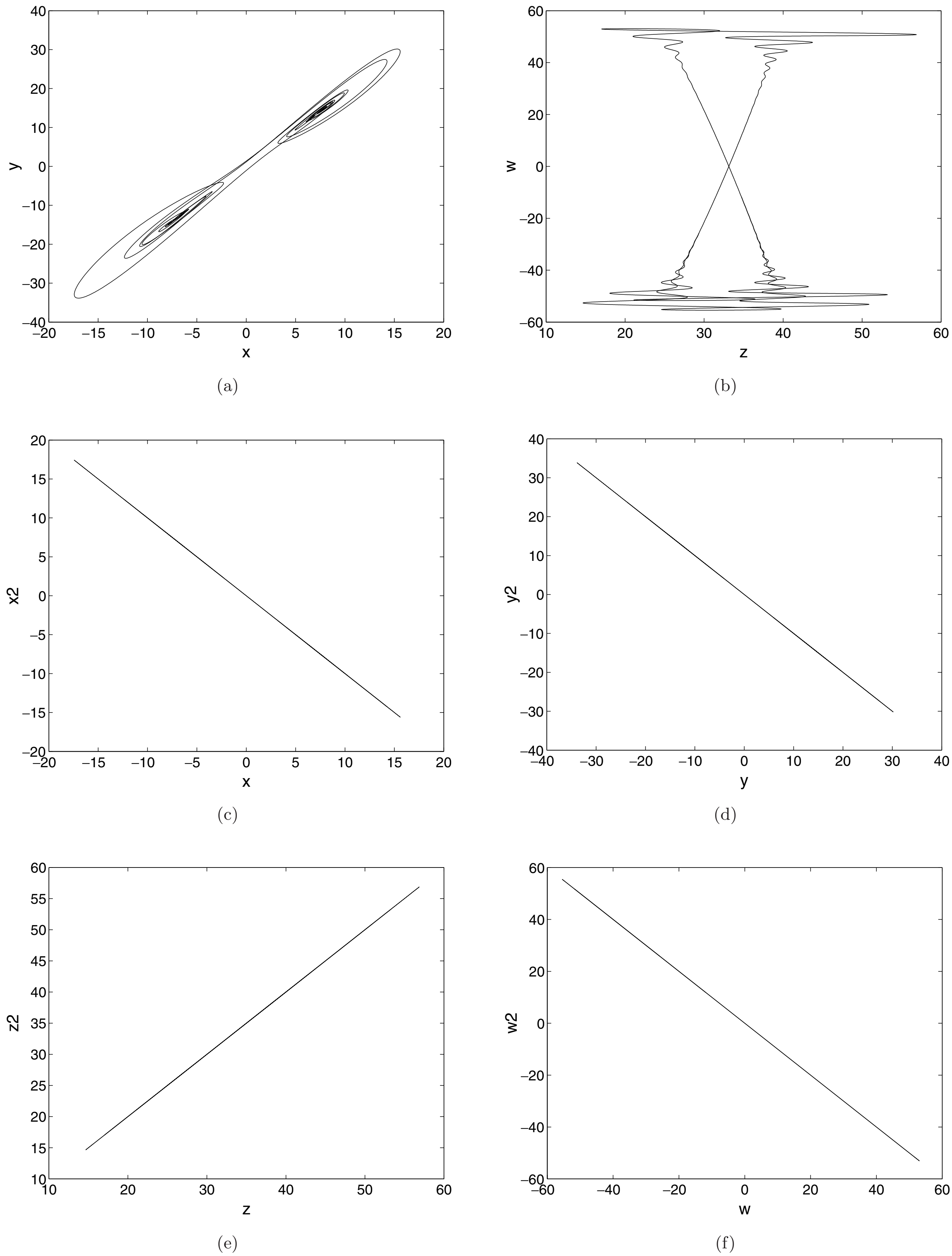


Fig. 4. The phase planes (x, y) and (z, w) of a coupled Chen system are shown in (a) and (b), and the plots showing complete synchronization for the variables x, y, z, w are (c), (d), (e) and (f) respectively.

is obvious that something happens in the coupled system and, in some sense, the dynamics become more uncorrelated implying lower sample times. If we take a look at the autocorrelation of x variable in Fig. 3, it is shown that the first zero crossing occurs for the uncoupled system. This result implies that the embedding dimension is lower in this case [Abarbanel *et al.*, 1993].

The third case takes place for a coupling strength value within the interval [15–30] approximately. Note that all the LEs are nonpositives and then the attractor must be nonchaotic [Grebogi *et al.*, 1984]. In Fig. 4, the phase planes (x, y) in (a) and (z, w) in (b) are shown for a coupling value $\varepsilon = 16.0$. The plots (c)–(f) show the variables from system 1 against those of system 2, thus showing that complete synchronization occurs for a coupling value of $\varepsilon = 16.0$.

Complete synchronization is achieved and chaos is suppressed. The chaos suppression in nonforcing coupled three-dimensional chaotic attractors was studied in [Bragard *et al.*, 2007] and the case for forced coupling was studied by Patidar *et al.* [2002]. Chaos suppression is expected because the linearized system converts the w variable in a simple time delay. This case could be compared with the chaotic attractor studied in [Boccaletti *et al.*, 2000].

Farther than coupling strength values beyond $\varepsilon = 30$, the largest LE becomes greater than zero and chaos returns again, but the system continues to be synchronized.

3. Lü System

The system studied in this work is another variation from Lorenz attractor [Lü & Chen, 2002] and extended to four dimensions, the hyperchaotic Lü system [Lü *et al.*, 2006], the equation set of which is:

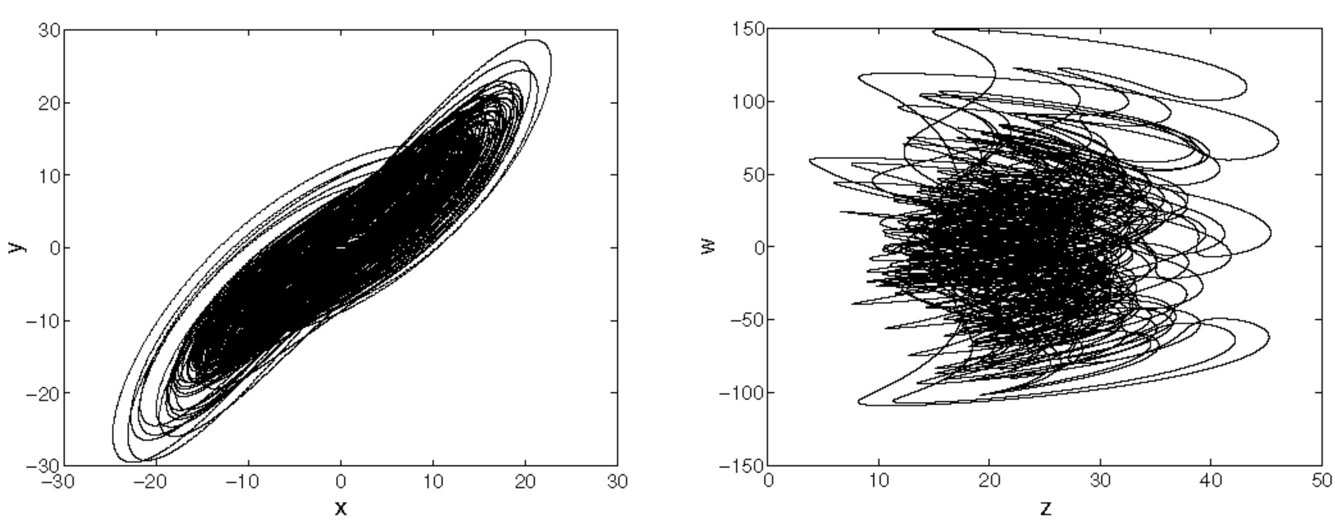
$$\begin{aligned}
x'_{1,2} &= a(x_{1,2} - y_{1,2}) + w + \frac{\varepsilon}{2}(x_{2,1} - x_{1,2}) \\
y'_{1,2} &= -x_{1,2}z_{1,2} + cy_{1,2} \\
z'_{1,2} &= x_{1,2}y_{1,2} - bz_{1,2} \\
w'_{1,2} &= x_{1,2}z_{1,2} - dw_{1,2}
\end{aligned} \tag{3}$$

As in the other cases, the variables are x, y, z and w, and the parameters are a, b, c and d. It looks very similar to the Chen system, but in this case there is a nonlinear term in the extended variable. In this work we set the parameter values as $a = 36$, $b = 3$, $c = 20$ and $d = 1$, in order to obtain an hyperchaotic attractor with two positive LE. The phase planes obtained with these values are plotted in Fig. 5.

As in the other cases studied, the fourth largest LE against the coupling factor is calculated and the results are shown in Fig. 6.

In this plot, it is possible to recognize some different parts that imply different dynamical behavior. As in the Chen System, the maximum of the largest LE does not correspond with the uncoupled system for the same reason.

Another interesting dynamical behavior occurs when the fourth largest LE becomes equal to zero.



(a) (b)

Fig. 5. The (x, y) phase plane in (a) and the (z, w) in (b) for an uncoupled Lü system.

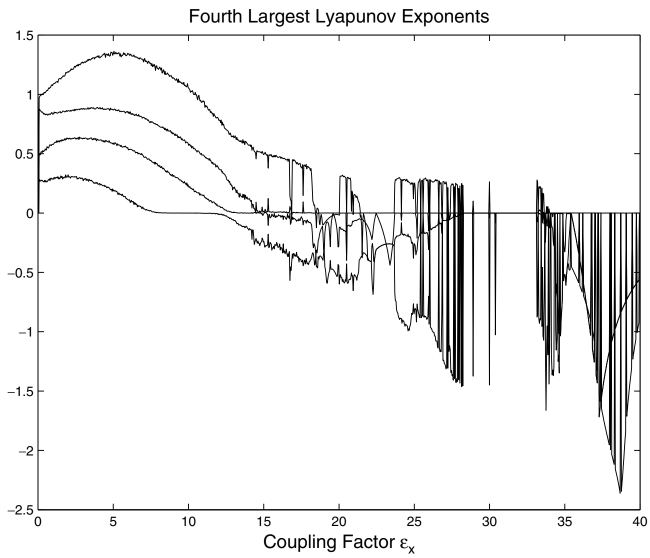


Fig. 6. LE against the coupling factor ε for a Lü hyperchaotic attractor. The lines going up and down reveal a riddled basin. For some initial conditions, the chaos is suppressed.

We relate this fact to the clearer phase planes now exhibited for the coupled Lü attractor. In order to check this fact, we have plotted the phase planes in Fig. 7 where complete synchronization is achieved.

Finally, some interesting results are shown corresponding to the interval where the four largest LE becomes equal to zero. This implies chaos suppression for coupling parameter values within [17–20] interval approximately. The phase planes are shown in Figs. 8(a) and 8(b). Figures 8(c)–8(f) show how the systems exhibit some correlation between them due to the closed shape that appears when the variables of one system are plotted against the variables of the other one.

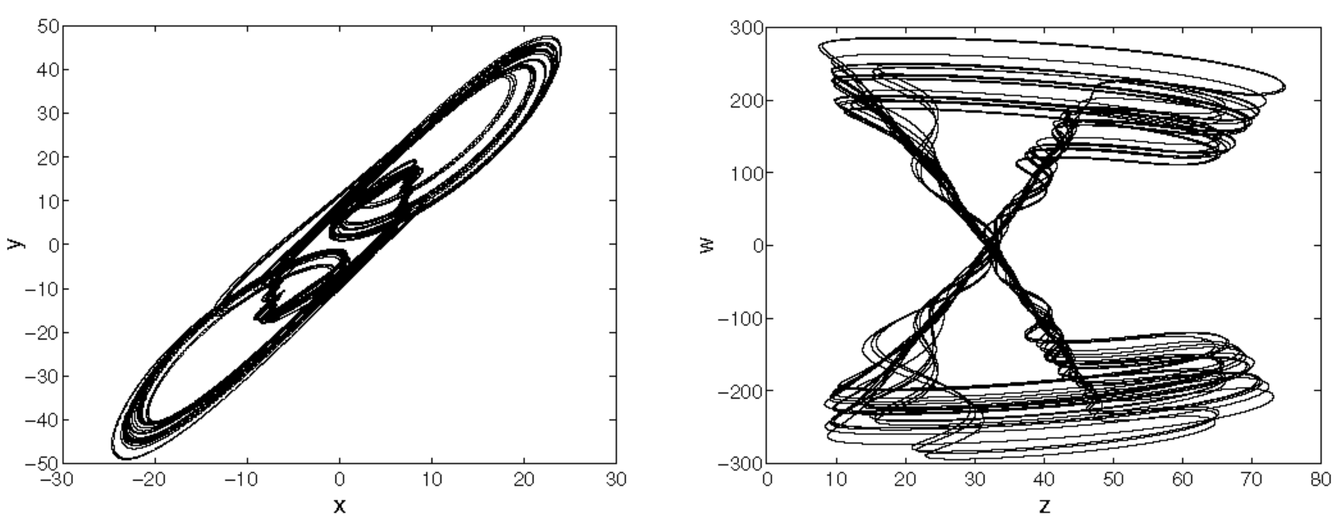
Note that this kind of closed shape in these plots proves that generalized synchronization appears [Boccaletti *et al.*, 2002], i.e. there exists a function that relates the variables between both systems. It means that a simpler attractor's dynamics does not imply a more restrictive synchronization. In this case, the coupling parameter transforms the hyperchaotic attractor in a strange nonchaotic attractor [Feudel *et al.*, 2006].

4. Takens–Bogdanov System

The equation set studied originally by D. Armbruster [Armbruster, 1990], needs a modification in order to calculate the LE [Vidal & Mancini, 2007]. Here, we show briefly this system, but if further information is required, a description on it and its synchronization features are available in [Vidal & Mancini, 2009]. The mathematical model implemented for the symmetrically coupled system is the following:

$$\begin{aligned} x'_{1,2} &= y_{1,2} + \frac{\varepsilon}{2}(x_{2,1} - x_{1,2}) \\ y'_{1,2} &= \mu x_{1,2} + x_{1,2}(a(x_{1,2}^2 + z_{1,2}^2) + bz_{1,2}^2) \\ z'_{1,2} &= w_{1,2} \\ w'_{1,2} &= \mu z_{1,2} + z_{1,2}(a(x_{1,2}^2 + z_{1,2}^2) + bx_{1,2}^2) \end{aligned} \tag{4}$$

Let x, y, z and w be the variables and a, b the parameters related to the physical and geometrical



(a) (b)

Fig. 7. Phases planes of the Lü system for $\varepsilon = 10$. (a) corresponds to the plane (x, y) and (b) to the (z, w).

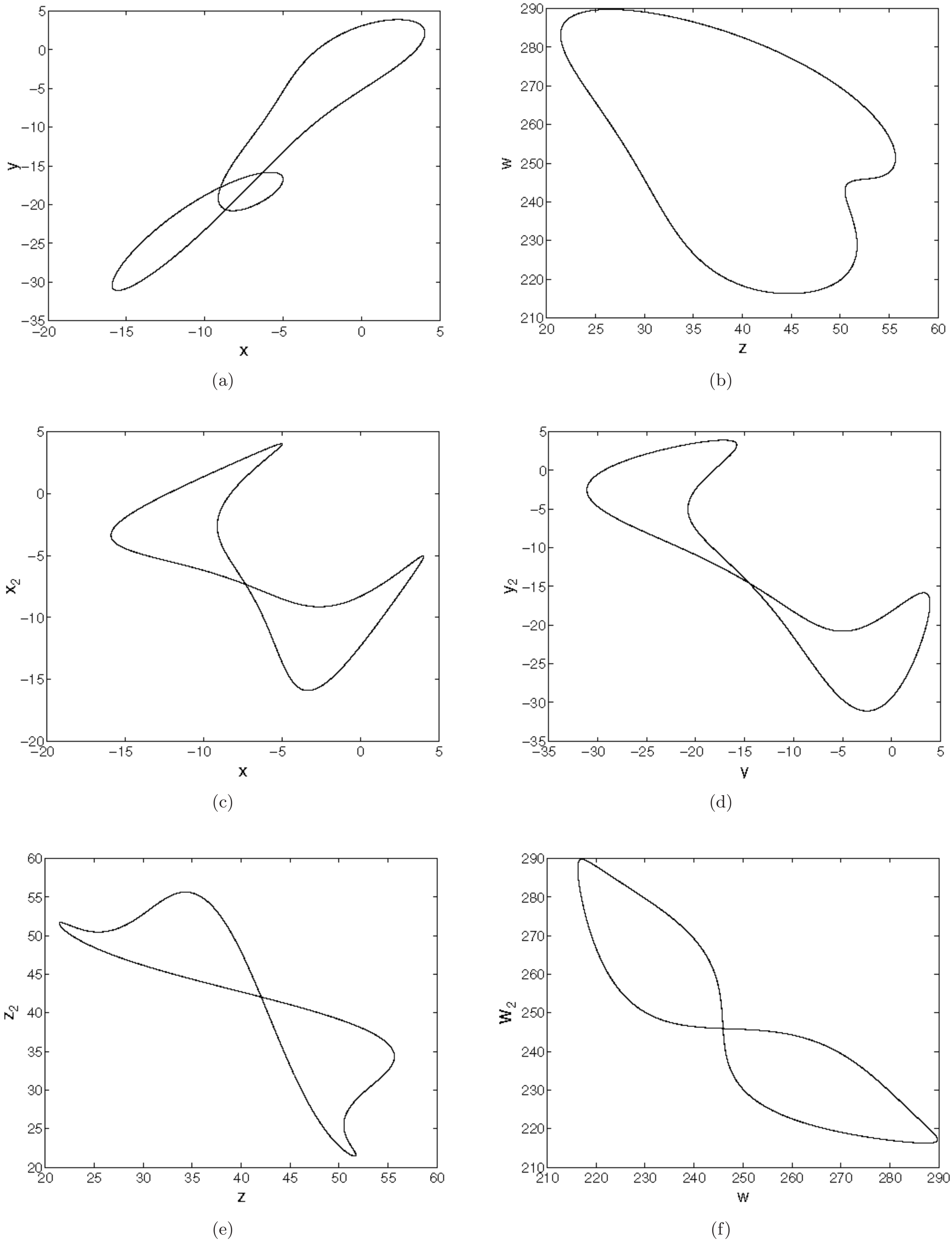
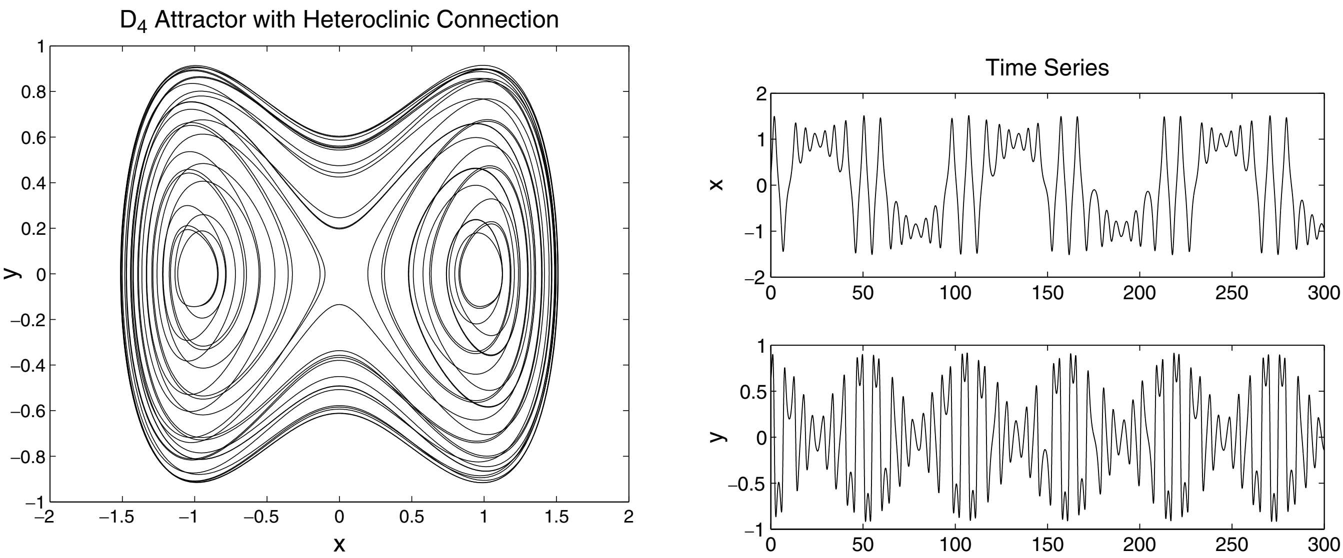


Fig. 8. Phase planes (x, y) and (z, w) are shown in (a) and (b) respectively. The plots which relate the variables x, y, z, w of one system against the other are shown in (c), (d), (e) and (f). In this case, the coupling value is $\varepsilon = 19.0$. Note that hyperchaos is suppressed and then generalized synchronization is achieved.



(a) (b)

Fig. 9. The (x, y) phase plane and the corresponding time series for the hyperchaotic Takens–Bogdanov attractor are shown in (a) and (b) respectively. Note that the heteroclinic connection provokes phase shifts between x and y on the time series. The phase plane (z, w) is omitted due to its similarity with (x, y).

properties of the experiment, and μ the control parameter. In a convection experiment, this parameter is related to the temperature. We have coupled the attractors through the x variable. So ε is the coupling strength parameter and the factor 1/2 is written in order to provide certain symmetry to the feedback signal.

The phase plane (x, y) for the uncoupled system is shown in Fig. 9 as a reference for comparing with the coupled cases, the phase plane (z, w) is omitted because it is very similar to the first one due to symmetries.

Also it is important to notice that these results look like a two Duffing's oscillators with an heteroclinic connection.

Before showing the LE of this system we have another important remark. This system has two positive and two negative eigenvalues with only two eigenvectors, so it is a degenerate saddle-node, with two instability directions becoming an hyperchaotic attractor in the sense defined by Rössler in [Rössler, 1979]. But we must be careful using the LE in this system. We have two positive and two negative LE but the zero exponent does not appear as expected [Haken, 1983]. In these kind of systems, LE cannot be calculated using methods without rescaling and reorthogonalization [Rangarajan *et al.*, 1998] and are not related directly to the entropy [Katok, 1980; Eckmann & Ruelle, 1985; Ott & Yorke, to be published]. In spite of this anomalous behavior, LE have been calculated to detect the synchronization regions as in the reported systems. To confirm the LE calculation, we have implemented three different algorithms [Sano & Sawada, 1985; Wolf *et al.*, 1985; Press *et al.*, 1992] obtaining similar results.

Once these remarks are done, we can show the LE against the coupling strength plot presented in Fig. 10.

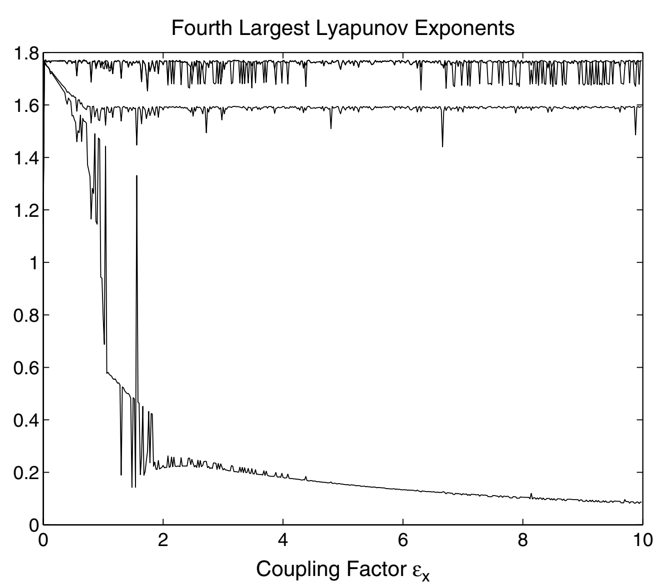
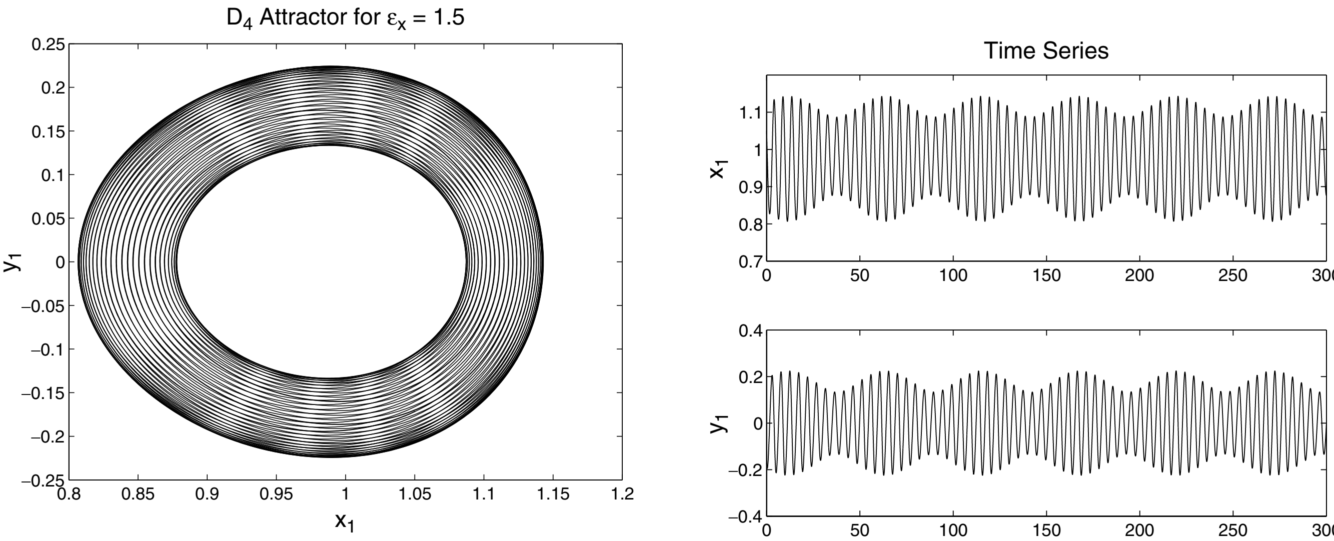


Fig. 10. The four largest LE against the coupling factor ε for the Takens–Bogdanov under D_4 symmetry attractor. The first and the second largest LE are almost equal.



(a) (b)

Fig. 11. Comparing the (x, y) phase planes (a) and the time series (b) of this figure against the uncoupled system, it is obvious that some reduction on the complexity occurs for some values of the coupling parameter. In this case, the coupling strength is $\varepsilon = 1.5$.

We define three regions in these figures by simple inspection: the first one corresponds to the interval $[0, 1.0]$ where the third and fourth largest LE decay, the next region is related to the interval $[1.0, 2.0]$ where there is a different decaying rate for the fourth largest LE. The last region is related to the plateau beyond the $\varepsilon = 2.0$ parameter value. On the other hand, the lines going up–down prove that a riddled basin [Alexander *et al.*, 1992] and a bubbling transition are involved [Venkataramani *et al.*, 1996]. Because the LE changes, the dynamics of the system also change depending on the initial conditions.

First, let the phase planes be shown in Fig. 11, and the time series corresponding to the coupling parameter values inside the LE valley.

Comparing these figures with the first it is obvious that the heteroclinic connection is destroyed by the coupling effect. Moreover, it is possible to define a phase in the time series signals using Hilbert Transform [Kantz & Schreiber, 1998] because it is easy to define a center where the trajectories spin around. These facts imply that the system becomes simpler than the uncoupled one.

In Fig. 12 is shown that the mean quadratic error tends to zero as time tends to infinity, thus there exists a complete synchronization between both attractors for a coupling value inside the second region defined above. Comparing Fig. 9 with the second region in Fig. 11 for a coupling factor $\varepsilon = 1.5$, it is obvious that there is a reduction on the complexity of the system.

Next step is to look what happens for a coupling parameter value outside the LE valley for a $\varepsilon = 10$. The plots that show these results are in Fig. 13.

In this case, the system looks simpler than the uncoupled case and a "quasi-synchronization" appears between x variables because $x_1 \approx x_2$ but $x \neq x_2$. On the other hand, it is not possible to

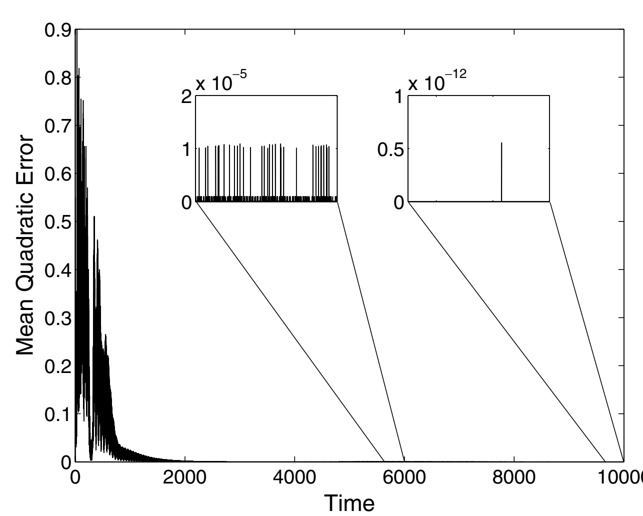


Fig. 12. The mean quadratic error for $\varepsilon = 0.75$ shows that complete synchronization is achieved. The insets show how the error decreases exponentially.

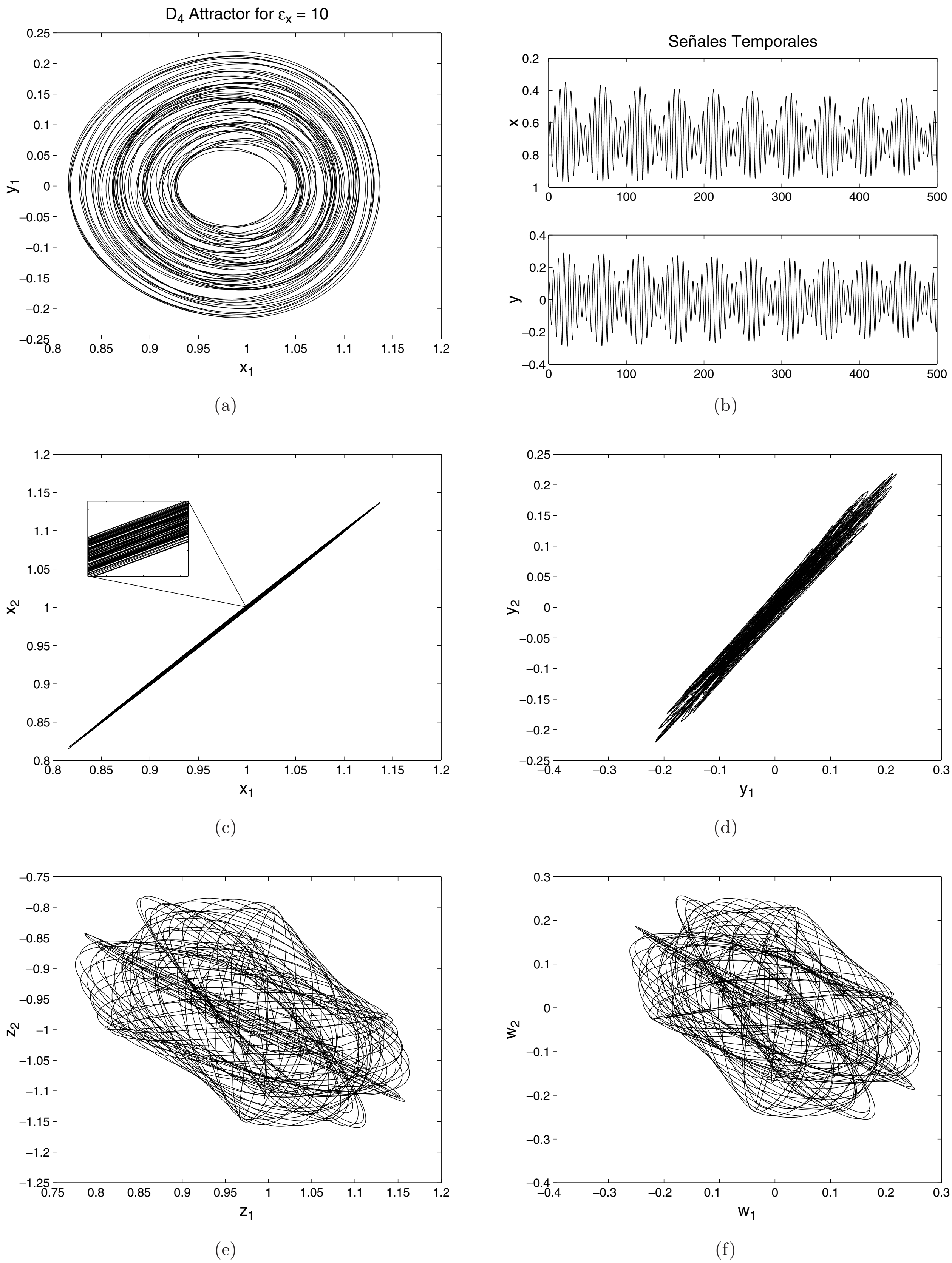


Fig. 13. The phases planes for the D_4 attractor (x, y) and the time series of these signals are shown in (a) and (b) respectively for a coupling strength parameter value of $\varepsilon = 10$. The synchronization plots for x, y, z and z are shown in (c), (d), (e) and (f) respectively. Note that an increase in the coupling strength does not imply a more restrictive synchronization, in this case the effect is the opposite, i.e. the loss of complete synchronization. Also note that the phase planes look more confusing than for the case of $\varepsilon = 1.5$.

define univocally a phase using the Hilbert's Transform as in the Fünnel attractor, a particular case of the Rössler attractor [Pikovsky *et al.*, 2001]. These new features suggest that the mutual information [Thomas & Cover, 1991] between both coupled systems decrease because the signals are spread on the phase planes and the systems become more uncorrelated than in the case when the coupling parameter is inside the second region.

5. Conclusions

We have calculated the four largest LE against the coupling factor through one variable between two identical hyperchaotic symmetrically coupled attractors.

It is shown that a window in the LE exists for all cases. Also, if the coupling strength is adjusted into this window the system's complexity is reduced and some kind of synchronization exists.

Furthermore, we conclude that an increase in the coupling strength does not imply a further decrease in the complexity of the system or a more restricting synchronization between the coupled pairs of identical oscillators.

On the other hand, chaos suppression exists for the extended hyperchaotic systems (Chen and Lü) but not for the Takens–Bogdanov hyperchaotic oscillator where the heteroclinic connection disappears.

Chaos suppression does not imply complete synchronization as for the Lü attractor. But sometimes could happen as in the hyperchaotic Chen attractor where complete synchronization is achieved by chaos suppression.

For the hyperchaotic Takens–Bogdanov and Lü attractors, complete synchronization is achieved, but in different regions of the LE against the coupling strength plot. For the Lü attractor, the window implies generalized synchronization.

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References

Abarbanel, H. D. I., Brown, R., Sidorowich, J. J. & Tsimring, L. S. [1993] "The analysis of observed chaotic data in physical systems," *Rev. Mod. Phys.* **65**, 1331–1392.

Alexander, J. C., Yorke, J. A., You, Z. & Kan, I. [1992] "Riddled basins," *Int. J. Bifurcation and Chaos* **2**, 795.

Armbruster, D. [1990] "Nonlinear evolution of spatiotemporal structures in dissipative continuous systems," *Codimension 2 Bifurcation in Binary Convection with Square Symmetry* (Plenum Press), pp. 385–398.

Boccaletti, S., Kurths, J., Osipov, G., Valladares, D. L. & Zhou, C. S. [2002] "The synchronization of chaotic systems," *Phys. Rep.* **366**, 1–101.

Boccaletti, S., Valladares, D. L., Kurths, J., Maza, D. & Mancini, H. [2000] "Synchronization of chaotic structurally nonequivalent systems," *Phys. Rev. E* **61**, 3712–3715.

Bragard, J., Vidal, G., Mancini, H., Mendoza, C. & Boccaletti, S. [2007] "Chaos suppression through asymmetric coupling," *Chaos* **17**.

Chen, S., Hu, J., Wang, C. & Lü, J. [2004] "Adaptive synchronization of uncertain Rössler hyperchaotic system based on parameter identification," *Phys. Lett. A* **321**, 50–55.

Eckmann, J. P. & Ruelle, D. [1985] "Ergodic theory of chaos and strange attractors," *Rev. Mod. Phys.* **57**, 617–656.

Feudel, U., Kuznetsov, S. & Pikovsky, A. [2006] *Strange Nonchaotic Attractors*: *Dynamics Between Order and Chaos in Quasiperiodically Forced Systems* (World Scientific).

Gao, T., Gu, Q. & Chen, Z. [2009] "Analysis of the hyper-chaos generated from Chen's system," *Chaos Solit. Fract.* **39**, 1849–1855.

Gao, T., Gu, Q., Chen, Z. & Yuan, Z. [2008] "A new hyper-chaos generated from generalized Lorenz system via nonlinear feedback," *Chaos Solit. Fract.* **35**, 390–397.

Gilmore, R. & Letellier, C. [2007] *The Symmetry of Chaos* (Oxford University Press).

Grebogi, C., Ott, E., Pelikan, S. & Yorke, J. A. [1984] "Strange attractors that are not chaotic," *Physica D* **13**, 261–268.

Haken, H. [1983] "At least one Lyapunov exponent vanishes if the trajectory of an attractor does not contain a fixed point," *Phys. Lett. A* **94**, 71–72.

Hoyle, R. [2006] *Pattern Formation*: *An Introduction to Methods* (Cambridge University Press).

Jia, Q. [2007] "Hyperchaos generated from the Lorenz chaotic system and its control," *Phys. Lett. A* **366**, 217–222.

Kantz, H. & Schreiber, T. [1998] *Nonlinear Time Series Analysis of Physiological Data* (Springer).

Katok, A. [1980] "Lyapunov exponents, entropy and periodic orbits for diffeomorphisms," *Publications Mathématiques de l'I.H.É.S.* **51**, 137–173.

Liu, Z., Li, Y. & Chen, G. [2007] "The basin of attraction of the Chen attractor," *Chaos Solit. Fract.* **34**, 1696–1703.

Lorenz, E. N. [1963] "Deterministic nonperiodic flow," *J. Atmos. Sci.* **20**, 130–141.

Lü, J. & Chen, G. [2002] "A new chaotic attractor coined," *Int. J. Bifurcation and Chaos* **12**, 659–661.

Lü, J., Yu, S., Chena, A. & Lua, J. [2006] "Generating hyperchaotic Lü attractor via state feedback control," *Phys. Lett. A* **364**, 103–110.

Matsumoto, T. [1984] "A chaottic attractor from Chua's circuit," *IEEE Trans. Circuits Syst.* **31**, 1055–1058.

Ondarçuhu, T., Mindlin, G. B., Mancini, H. L. & Pérez-García, C. [1993] "Dynamical patterns in Bénard–Marangoni convection in a square container," *Phys. Rev. Lett.* **70**, 3892–3895.

Ott, W. & Yorke, J. A. [to be published] "When Lyapunov exponents fail to exist," to be published in *Phys. Rev. E.*

Patidar, V., Sud, K. & Pareek, N. K. [2002] "Suppression of chaos using mutual coupling," *Phys. Lett. A* **304**, 121–129.

Pikovsky, A., Rosenblum, M. & Kurths, J. [2001] *Synchronization: A Universal Concept in Nonlinear Science* (Cambridge University Press).

Press, W., Teukolsky, S., Vetterling, W. & Flannery, B. [1992] *Numerical Recipies in C: The Art of Scientific Computing* (Cambridge University Press).

Rangarajan, G., Habib, S. & Ryne, R. D. [1998] "Lyapunov exponents without rescaling and reorthogonalization," *Phys. Rev. Lett.* **80**, 3747–3750.

Rössler, O. E. [1976] "An equation for continuous chaos," *Phys. Lett. A* **57**.

Rössler, O. E. [1979] "An equation for hyperchaos," *Phys. Rev. A* **71**, 155–157.

Sano, M. & Sawada, Y. [1985] "Measurement of the Lyapunov spectrum from a chaotic time series," *Phys. Rev. Lett.* **55**, 1082–1085.

Thomas, J. A. & Cover, T. M. [1991] *Elements of Information Theory* (Wiley-Interscience).

Venkataramani, S. C., Hunt, B. R. & Ott, E. [1996] "Bubbling transition," *Phys. Rev. E* **54**, 1346–1360.

Vidal, G. & Mancini, H. L. [2007] "Complete synchronization between hyperchaotic space-time attractors," *Proc. New Trends and Tools in Complex Networks*, eds. Criado, R., Pello, J. & Romance, pp. 125–134.

Vidal, G. & Mancini, H. L. [2009] "Hyperchaotic synchronization under square symmetry," *Int. J. Bifurcation and Chaos* **19**, 719–726.

Wolf, A., Swift, J. B., Swinney, H. L. & Vastano, J. A. [1985] "Determining Lyapunov exponents from a time series," *Physica D* **16**, 285–317.

Zhou, T., Tang, Y. & Chen, G. [2004] "Chen's attractor exists," *Int. J. Bifurcation and Chaos* **14**, 3167–3177.