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HYPERCHAOTIC SYNCHRONIZATION UNDER SQUARE SYMMETRY

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We study two identical hyperchaotic oscillators symmetrically coupled. Each oscillator represents a codimension-2 Takens–Bogdanov bifurcation under square symmetry, and was used to model a convection experiment in a time-dependent state. In the coupled system, the Lyapunov exponents behavior against the coupling parameter is used to detect changes in the dynamics, and the synchronization state is controlled by checking the phase planes. Complete synchronization is achieved without chaos suppression in a coupling parameter interval. Outside this window, complete synchronization cannot be generally achieved. As a consequence of a bubbling transition, synchronization is obtained only for some particular values of the initial conditions.

Keywords: Hyperchaotic attractors; Takens–Bogdanov bifurcation; synchronization; Lyapunov exponent.

1. Introduction

Since the Pecora and Carroll paper in 1990 [Pecora & Carroll, 1990], chaos synchronization has been developed as a very active field in science. Different regimes of synchronization [Boccaletti *et al.*, 2001] have been reported in almost all the classical models of low dimensional chaotic systems in nonlinear dynamics [Lorenz, 1963; Rössler, 1976; Matsumoto, 1984], with important applications in many fields from basic physics to electronic circuits, engineering, biology or economy [Pikovsky *et al.*, 2001; Boccaletti *et al.*, 2002].

Moreover, coupled maps [De San Roman *et al.*, 1998], model equations like Ginzburg–Landau or delayed systems [Boccaletti *et al.*, 2000] have been shown to be synchronized, both between identical or nonidentical systems, and under symmetric or asymmetric coupling [Boccaletti *et al.*, 1999; Bragard *et al.*, 2003]. Experimental evidence of synchronization in a chaotic convective flow [Maza *et al.*, 2000] and different experiments with electronic circuits was reported and can be found in reviews [Pikovsky *et al.*, 2001; Boccaletti *et al.*, 2002]. More recently, a new field of applications related to synchronization in complex network dynamics is growing [Boccaletti *et al.*, 2006].

Hyperchaotic behavior appears in a nonlinear dynamical system when more than one Lyapunov exponent becomes positive [Rössler, 1979] and normally arises as a natural regime in extended space-time systems, delayed systems or in situations where many oscillators are coupled (a normal situation in complex networks). In all these cases, it is usually very difficult to understand what is happening physically inside the system for different reasons, like the presence of spatial symmetries restricting the possible solutions, or delays transforming the system into an infinite-dimensional one.

If the number of problems analyzed on synchronization in low dimensional chaotic systems can be considered very high, numerical works reporting the behavior of hyperchaotic systems number much lower. This work focuses on the analysis of synchronization in a mathematical model [Armbruster, 1990] representing a bifurcation sequence in a codimension-2 point (two modes bifurcating simultaneously) with square symmetry. This model was used before to represent a convection experiment in small square box as reported in [Ondarçuhu *et al.*, 1993; Mindlin *et al.*, 1994; Ondarçuhu *et al.*, 1994].

The mathematical model describing the situation observed in the experiment contains four variables (x, y, z, w) and nine parameters. In this work, we present the results of synchronization between two hyperchaotic identical systems, guided by the symmetries of the system. We have used the temporal signal outputs in order to choose the variables to be synchronized and to couple them bidirectionally.

In a recent paper [Bragard *et al.*, 2007] we reported an analysis of several low dimensional systems on which windows of synchronization appear suppressing the chaotic oscillations. In that paper, windows of synchronization were detected by considering the Lyapunov exponent as a function of the coupling strength. In this work we have used the same method to detect the synchronized state window and the phase space representation to analyze the dynamic state of the coupled system. The mathematical model of the coupled system is presented in Sec. 2 and the synchronization process in Sec. 3. Finally, the results and conclusions are presented in Secs. 4 and 5.

2. Mathematical Model for the Hyperchaotic Attractor

This work is based on a mathematical model for codimension-2 point bifurcation with double zero eigenvalues (Takens–Bogdanov bifurcation) under square symmetry, which shows hyperchaotic behavior. This model was first studied by Armbruster [1990] and was used to model experiments cited in [Ondarçuhu *et al.*, 1994, 1993; Mindlin *et al.*, 1994]. The equation set is:

$$x' = y + \gamma^2 fz(zy - wx) \tag{1}$$

$$\begin{aligned} y' = \mu x + x(a(x^2+z^2) + bz^2) \\ + \gamma(\nu y + y(c(x^2+z^2) + ez^2 \\ + dx(xy + zw))) + \gamma^2 fw(zy - wx) \end{aligned} \tag{2}$$

$$z' = w - \gamma^2 fx(zy - wx) \tag{3}$$

$$\begin{aligned} w' = \mu z + z(a(x^2+z^2) + bx^2) \\ + \gamma(\nu w + w(c(x^2+z^2) + ex^2 \\ + dz(xy + zw))) - \gamma^2 fy(zy - wx) \end{aligned} \tag{4}$$

where ν and μ are the unfolding parameters, γ is a scaling parameter, and a, b, c, d, e, f are order one parameters. We have shown in [Vidal & Mancini, 2007] that this dynamical system has a limitation in numerical simulations for some values of γ, due to the relation between this parameter and dissipativity coefficient. If $\gamma \neq 0$, when a long time series calculation is performed the series explode (when $\gamma > 0$) or become dissipative (when $\gamma < 0$). Consequently, under this condition the connection disappears between the numerical results and the physical behavior. Then, the value of $\gamma \approx 0$ must be forced in order to impose that the system becomes conservative and stable. This is not a strong constraint in the experiment because γ is related to the time-dependent part of the total dissipativity of the system (less than 10%) [Vidal & Mancini, 2007].

Accepting $\gamma = 0$ for long time series, the following simplified equation set is obtained:

$$x' = y \tag{5}$$

$$y' = \mu x + x(a(x^2+z^2) + bz^2) \tag{6}$$

$$z' = w \tag{7}$$

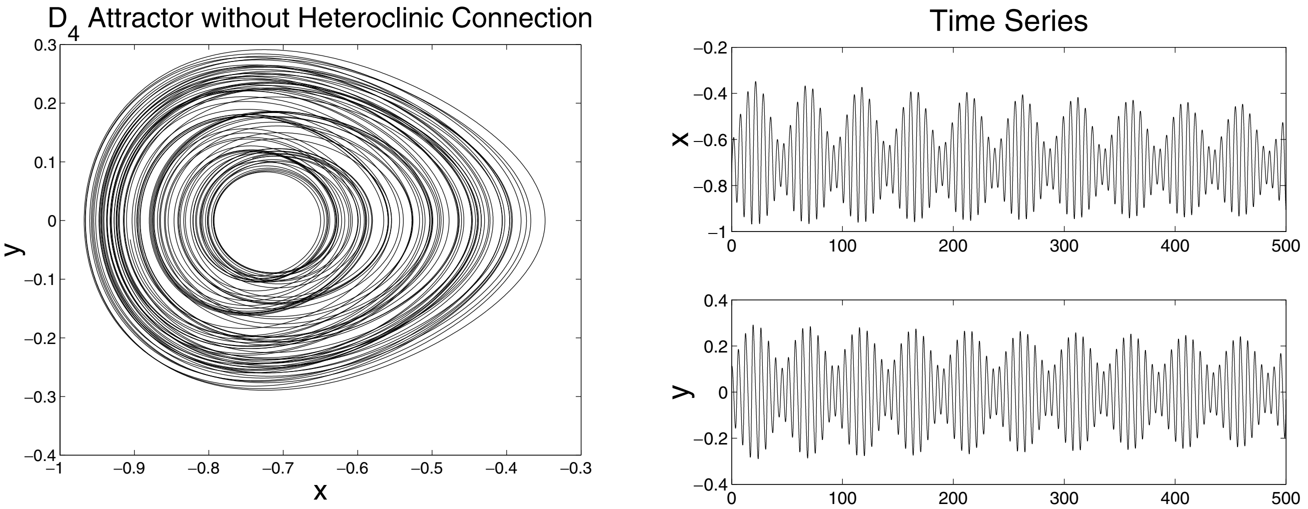
$$w' = \mu z + z(a(x^2+z^2) + bx^2) \tag{8}$$

where a, b and μ are parameters related to physical characteristics of the fluid and geometrical properties of the container.

Choosing the values for these parameters properly the system shows very different dynamics. Fixing a and b, it is possible to use μ as a control parameter for selecting the dynamics.

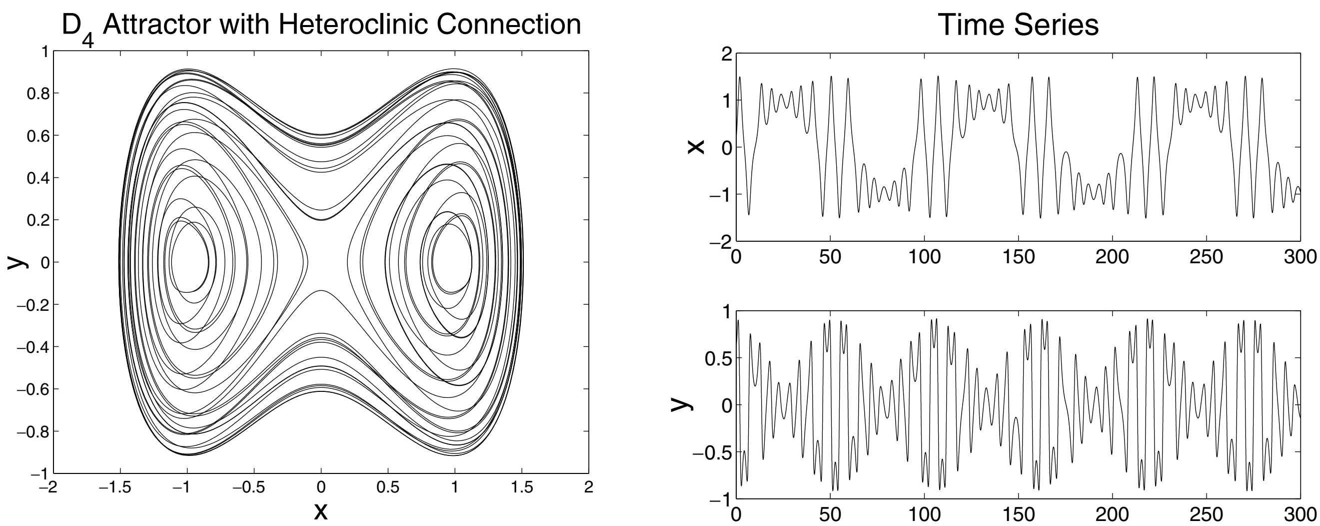
We have checked the results comparing the bifurcation sequence and the phase planes with the results obtained in other works [Armbruster, 1990; Ondarçuhu *et al.*, 1994, 1993; Mindlin *et al.*, 1994]. In order to perform the numerical simulations we have used the fourth order Runge–Kutta method with $\Delta t = 10^{-2}$.

As it was described in [Ondarçuhu *et al.*, 1994] the bifurcation process begins in a double zero point related to the original patterns imposed by the boundary conditions. By increasing μ the spatial symmetry is broken and the system selects one of the two possibilities that preserves diagonal symmetries. A further increase in μ leads the system to become time-dependent displaying the chaotic behavior plotted in Fig. 1. The heteroclinic connection appears linking both sides of the phase plane composed by two Duffing's oscillators, as was



(a) (b)

Fig. 1. (a) The phase plane (x, y) for $\mathbb{D}_4$ attractor before the heteroclinic connection. The time series of (x, y) is shown in (b). The (z, w) phase plane and time series are omitted because they are very similar to the first one.



(a) (b)

Fig. 2. Time series and (x, y) phase plane are shown in (a) and (b) respectively. Note that the heteroclinic connection provokes phase shifts between x and y on the time series.

remarked in [Guckenheimer & Holmes, 1983]. The temporal behavior in that condition can be seen in Fig. 2.

Let us now explain something else about the symmetry properties appearing in this model. These properties comes from the "dihedric" group $\mathbb{D}_4$, that represents the rotation and reflections for a square geometry:

$$\begin{aligned} \tau &: (x, y, z, w) \rightarrow (z, w, x, y) \\ \rho &: (x, y, z, w) \rightarrow (-x, -y, z, w) \end{aligned} \tag{9}$$

The symmetries of the square box are reflected in Eqs. (1)–(4). These symmetry transformations present two very similar phase planes (x, y) and (z, w).

3. Synchronization Process

In this section, we present the coupling and the calculation technique used. Firstly, let us describe the scenario for free oscillations before reporting the coupling effects. Choosing suitable parameter values to set the system for the heteroclinic connection, results in a hyperchaotic attractor with the temporal outputs plotted in Fig. 2.

By using the symmetry properties, it is possible to simplify the system until two free variables: on the one hand there is a function which describes the (x, z) plane, on the other hand there is the function for (x, y) and the (z, w) phase planes. So, there are two degrees of freedom represented by x and z variables which are related by a nonlinear coupling, the other variables are just a function of these.

Assuming that we connect one of these variables, the coupled system has to reduce the complexity, because it reproduces the same symmetry in both systems. We also expect that the simplest coupling effect will be similar to that shown in [Bragard *et al.*, 2007] i.e. a valley in the Lyapunov exponents where complete synchronization [Pikovsky *et al.*, 2001] is achieved due to a complexity reduction. We confirm this statement in the next section by the numerical simulation results.

In this work a coupling between the variables x of each attractor, and the parameter ε as a coupling strength factor are used. This is a symmetric coupling method instead of the unidirectional coupling used in [Chen *et al.*, 2004; Jia, 2007; Zou, 2005; Wang & Liu, 2006]. By using this method we are balancing both systems instead of forcing one system to follow the other, as in master-slave configuration.

The equation set for this coupling method is:

$$\begin{aligned}
x_1' &= y_1 + \frac{\varepsilon}{2}(x_2 - x_1)\\
y_1' &= \mu x_1 + x_1(a(x_1^2 + z_1^2) + bz_1^2)\\
z_1' &= w_1\\
w_1' &= \mu z_1 + z_1(a(x_1^2 + z_1^2) + bx_1^2)\\
x_2' &= y_2 + \frac{\varepsilon}{2}(x_1 - x_2)\\
y_2' &= \mu x_2 + x_2(a(x_2^2 + z_2^2) + bz_2^2)\\
z_2' &= w_2\\
w_2' &= \mu z_2 + z_2(a(x_2^2 + z_2^2) + bx_2^2)
\end{aligned} \tag{10}$$

For the purpose of this work, we have swept the coupling parameter ε within the interval $[0, 10]$.

To analyze the different synchronization states we use the Lyapunov exponents graph, as was done in [Bragard *et al.*, 2007], instead of calculating the transversal Lyapunov exponents as in [Maistrenko *et al.*, 1998; Galias, 1999].

Calculations are performed by integrating the eight linearized systems from the two original ones. For these systems, we have used the Runge–Kutta method with the same Δt, calculating the exponents in $5 \cdot 10^6$ time steps. The Gramm–Schmidt normalization process was done every 50 time steps, and this operation requires the most computational time. We begin to calculate the Lyapunov exponents after 10^6 time steps. The number of time steps to be discarded depends on the volume where our initial conditions are randomly varied. We have verified that this transitory is sufficient for a good accuracy in calculations.

In order to keep the temporal evolution of the system inside the attractor, it is necessary to limit the random initial condition inside a sufficiently small hypercube centered at the origin.

4. Results

Figure 3 shows the Lyapunov exponents against the coupling factor ε changing in the interval $[0, 10]$. There is a single run for the Lyapunov exponents. Due to the existence of a bubbling transition and a riddle basin [Venkataramani *et al.*, 1996] some spikes appear in the Lyapunov plot.

In the figure, it can be seen that there is an interval for ε values where the Lyapunov exponents

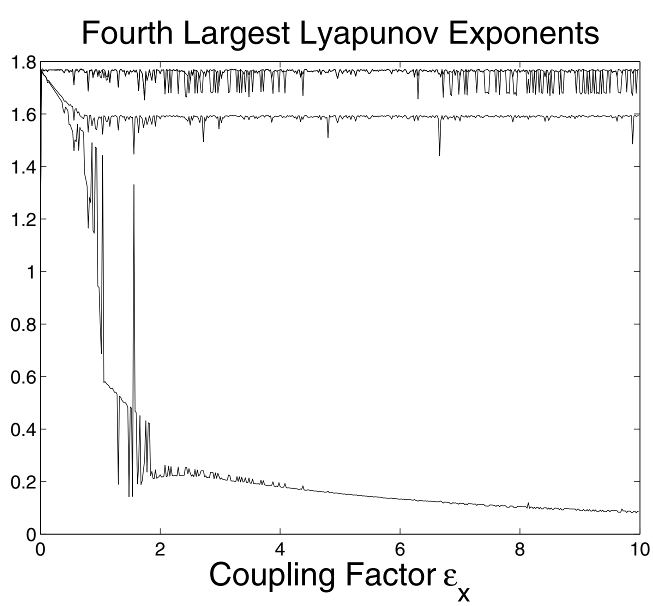


Fig. 3. The four positive Lyapunov exponents for a coupling factor ε between $[0, 10]$. The two largest Lyapunov exponents are overlapped.

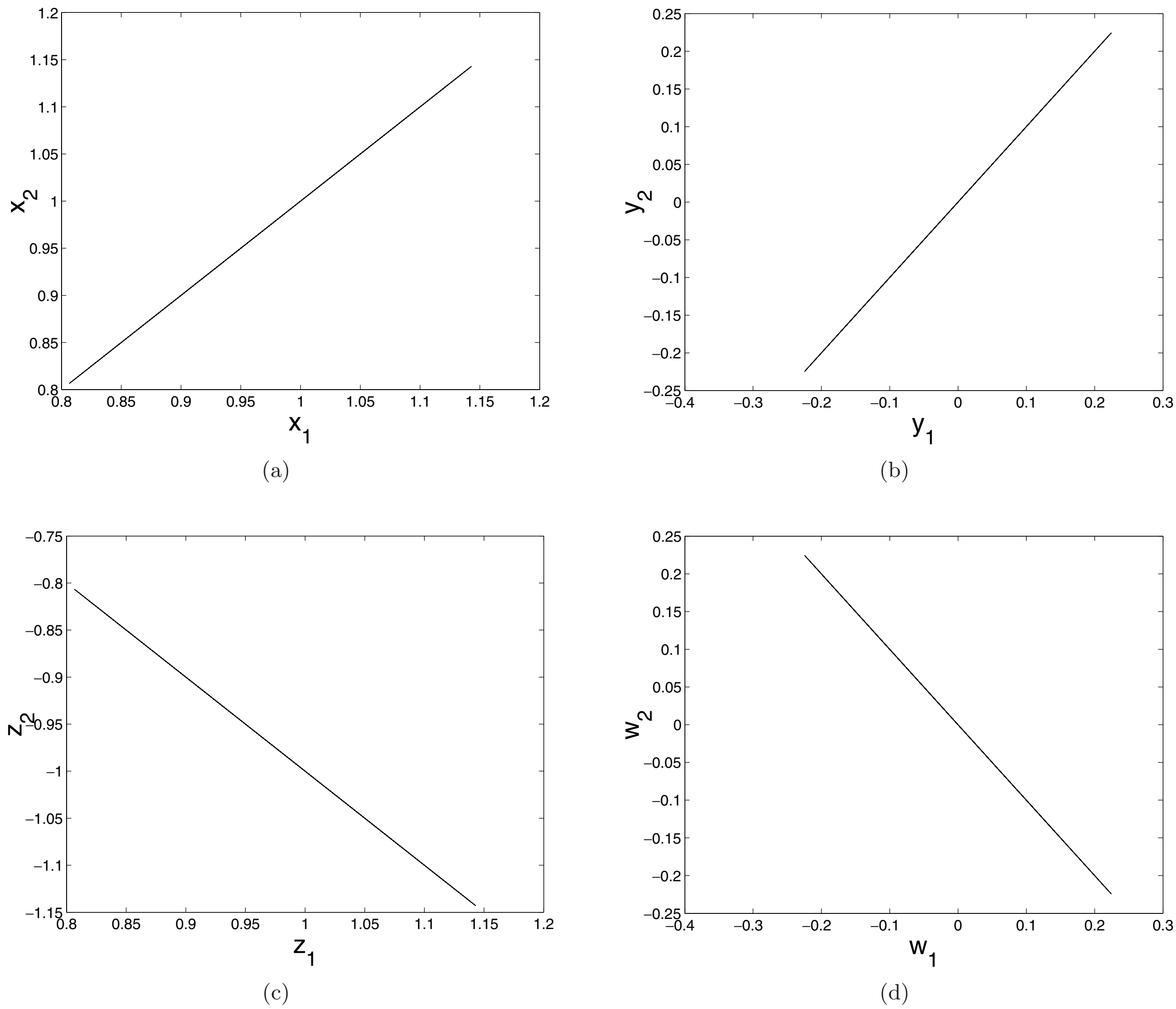


Fig. 4. Complete synchronization is achieved between both systems. We show the synchronization plots for x, y, z, w variables in (a)–(d) respectively for a coupling factor $\varepsilon = 1.5$.

decrease suddenly. This can be compared with the results shown in [Bragard *et al.*, 2007] where two identical Rössler, Lorenz and Lotka–Volterra attractors were coupled showing similar behavior in the exponents graph. The valley appearing in those plots corresponds to the values where complete synchronization was achieved.

Figure 4 shows the complete synchronization state by plotting system 2 against system 1 variables. As a strong contrast to the low dimensional synchronization considered in [Bragard *et al.*, 2007], in our work, complete synchronization is achieved but without chaos suppression inside the Lyapunov exponents valley.

In Fig. 6 we show the phase plane (x, y) in (a) and a Poincaré section in (b). Evidently the plot (a)

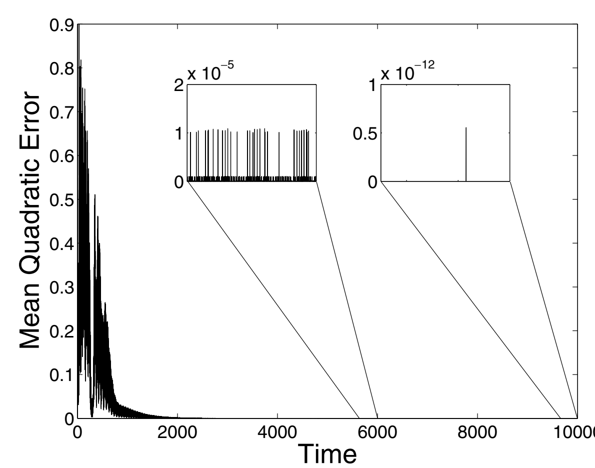


Fig. 5. Synchronization mean quadratic error for $\varepsilon = 1.5$. Note that the insets show clearly that the errors converge to zero.

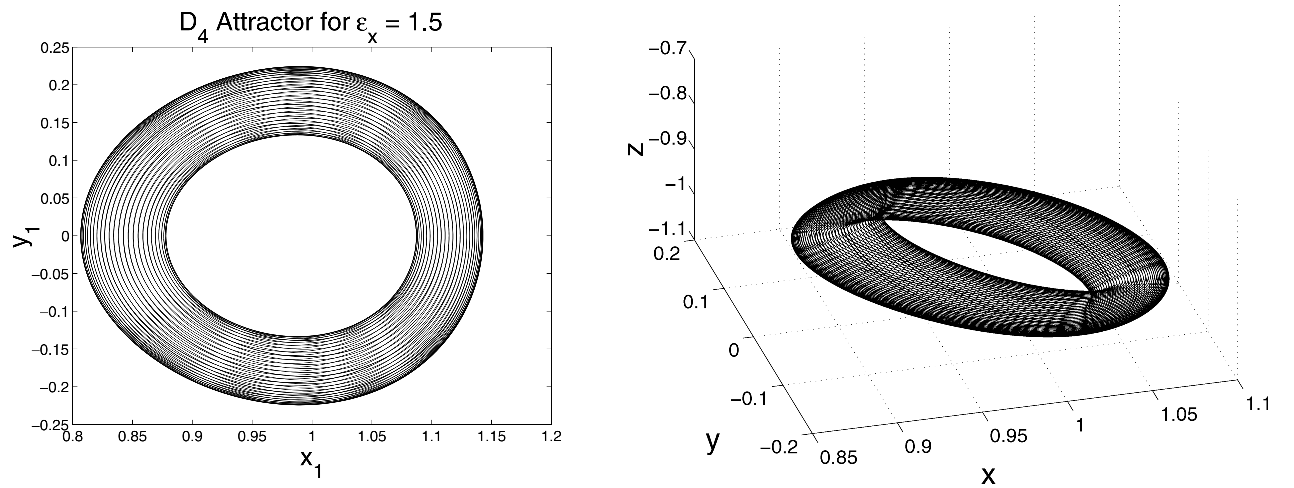
seems a quasi-periodic attractor, but the Poincaré section shows clearly that it is not [Strogratz, 1994]. In order to clarify even more that synchronization is actually achieved for any realization with randomly chosen initial conditions, we have calculated the mean quadratic error of the synchronization as follows:

$$\mathrm{MQE} = \sqrt{\frac{(x_1 - x_2)^2 + \cdots + (w_1 - w_2)^2}{4}} \tag{11}$$

Also, we have plotted the evolution of this measure against time in Fig. 5.

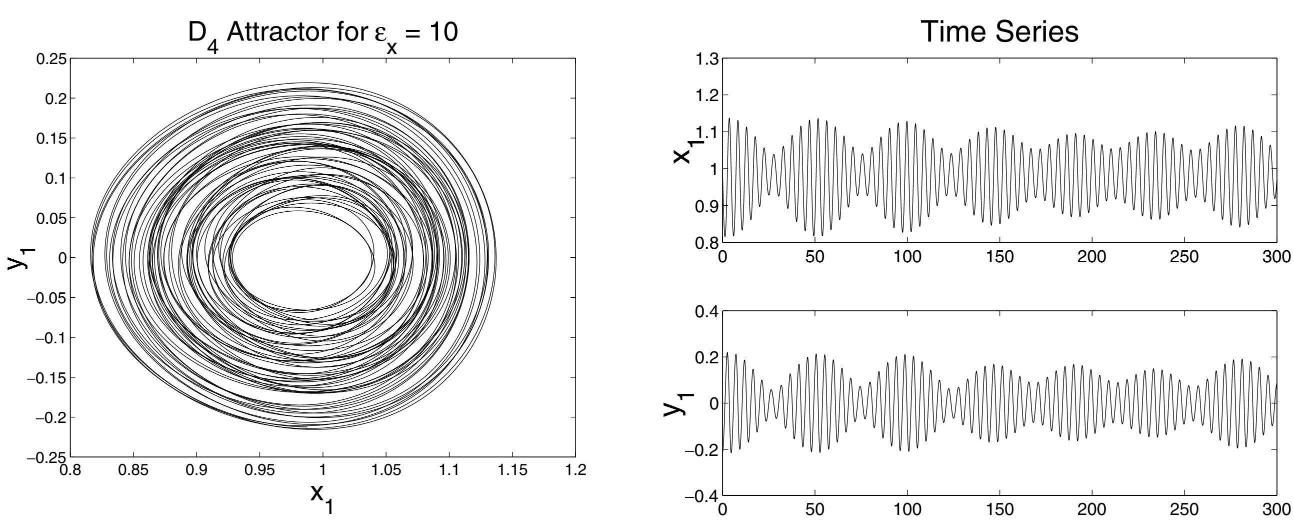
Another interesting difference appears when the phase planes are compared for different ε values inside and outside the valley. For the last condition, the attractors become more complex as it can be seen when comparing Figs. 6 and 7. Clearly the system in Fig. 6 has a more regular dynamics.

It is important to remark that a stronger coupling factor does not imply a better correlation in the time series, in other words, more coupling does not imply a better synchronization. This fact is evident in Fig. 8. It must be remarked that in the section (a) complete synchronization between x_1 and



(a) (b)

Fig. 6. Phase planes (x_1, y_1) in (a). Note that this attractor for a coupling factor $\varepsilon = 1.5$ is a simplified version of the original one presented in Fig. 1. A Poincaré section with (x, y, z) variables is shown in (b) for $\varepsilon = 1.5$.



(a) (b)

Fig. 7. The (x_1, y_1) phase plane for $\varepsilon = 10$ value is shown in (a) plot. Also the time series are shown in (b) for the same parameter value.

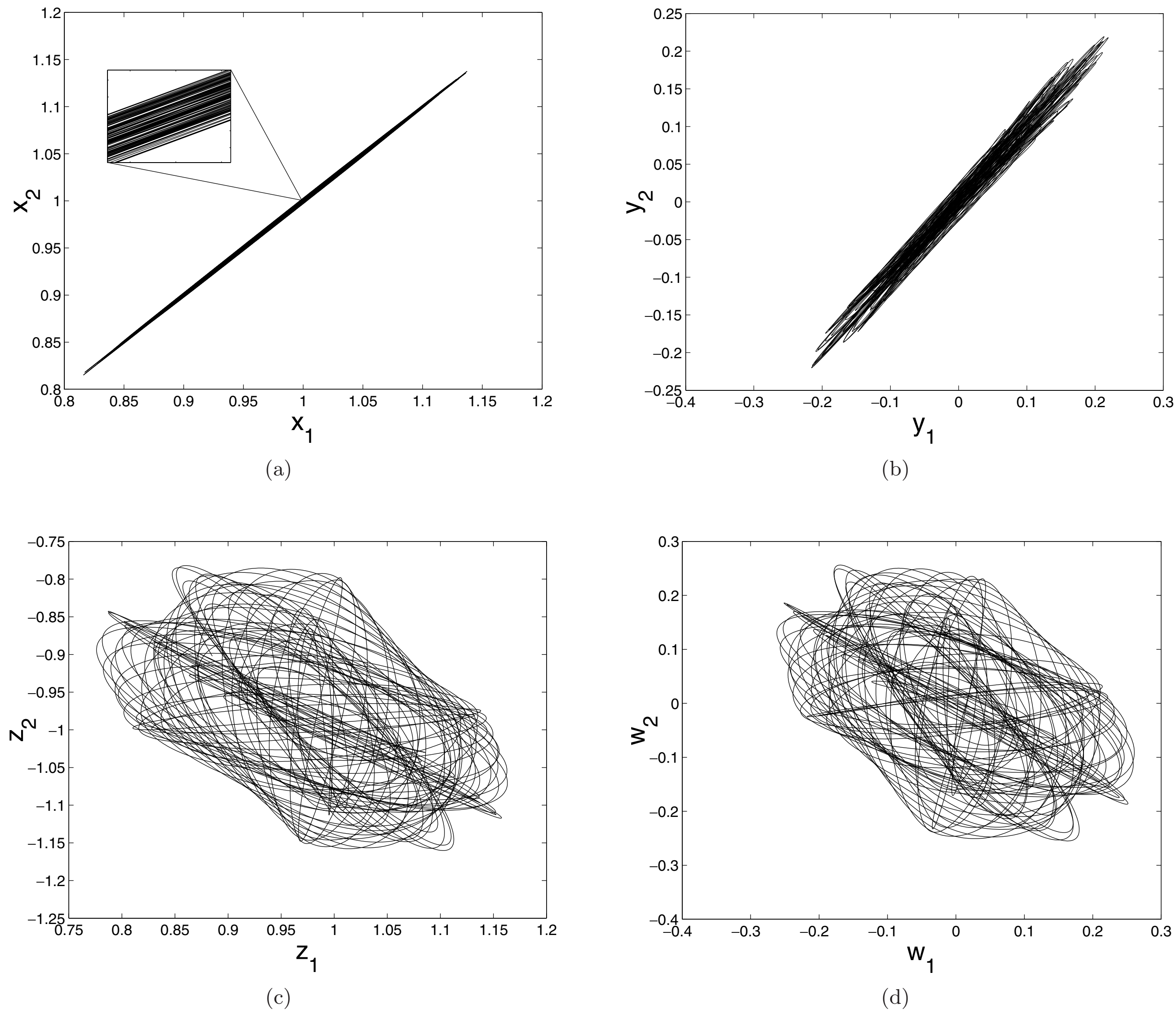


Fig. 8. Variables from system 1 against the variables from system 2. The plot (a) looks like a quasi-synchronization, but the zoom shows that there is no complete synchronization; in the other cases (b)–(d) it is clearly shown.

x_2 is not achieved because $x_1 \neq x_2$. The plot looks like a straight line but it is not completely thin, appearing like a "quasi-synchronized" state.

5. Conclusions

This study on hyperchaotic synchronization reports an unnoticed behavior so far. As in previously reported low dimensional case [Bragard *et al.*, 2007], a valley exists in the plot of Lyapunov exponents against coupling factor where complete synchronization is achieved and the complexity is reduced.

But this paper shows that chaos is not suppressed in hyperchaotic attractors. We observe that chaos suppression comes from complexity reduction in the coupled system. This reduction of complexity can also be observed in high dimensional systems obtained by delays as seen in the results presented in [Boccaletti *et al.*, 2000].

A further additional increase in the coupling coefficient bringing the coefficient out of the Lyapunov exponents valley, leads to an additional reduction of complexity that brings the system to more complex situations.

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