

Complete Synchronization between Hyperchaotic Space-Time Attractors

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Abstract

Many dynamical processes that occur in nature or in experiments are space-time bifurcations in systems with symmetries. One of the most outstanding examples of such bifurcations is the Takens-Bogdanov bifurcation [1] that has been used to model Codimension-two bifurcations with double zero eigenvalues and square symmetries, that exhibit chaotic behavior [2, 3]. In particular, the dynamics of termo-convective experiments in square cells with a small aspect ratio have been modeled with the groups of symmetry Z_2 [4] and D_4 [5].

In the first part of this work we describe the dynamic behavior appearing in the equation system with the group of symmetry D_4 proposed in [2], as a function of the different parameters. The space of parameters is analyzed in order to identify those variables that could be useful to synchronize, in space and time, two identical systems of this kind. In the second part we describe the first results obtained to synchronize two identical systems.

Keywords: NonLinear Dynamics, Benard-Marangoni Convection.

MSC 2000:

1. Introduction

Bifurcations, in presence of symmetries appearing in extended systems with patterns formed by more or less periodic structures, have a symmetry group directly related to the lattice symmetry of the periodic structure. Under this conditions, solutions in bifurcations are normally very complex and have been

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found in detail only in some particular situations. But in strongly confined systems where natural structures are imposed by boundary conditions with less symmetries, some typical situations of this kind of bifurcations can be analyzed more easily. There is a classical situation that appear in Benard-Marangoni convection [6] when the aspect ratio Γ (the ratio between the medium horizontal dimension D divided by depth of the fluid d ($\Gamma = D/d$)) is low enough as to put the system near a Codimension-2 point. A Codimension 2 point in a convective system let us analyse experimentally this kind of bifurcation in high detail and to compare it with theoretical and numerical solutions [1].

Codimension two points in small aspect ratio convective systems are related to the simultaneous instability of two convective modes. Unfortunately, only in a few cases, this kind of bifurcations have been studied experimentally [4, 5, 7]. The systems cited in these references (considering square and cylindrical symmetries) have been studied experimentally and also have been modeled by solving the Navier-Stokes equations and using adequate lateral boundaries conditions. This works consider resonances between the lowest order modes (1:1, 1:2), [8, 9]. Numerical simulations reproduce qualitatively the complex dynamical results obtained experimentally as a function of the control parameter.

But in [4], it was shown that the dynamic behavior can be modeled by a Takens-Bogdanov bifurcation reproducing all the sequence of bifurcations without reference to the physical variables (the velocity and temperature fields). In this work, the vector dynamics of the system was represented by a scalar (geometrical) variable, easily observed and measured in the planform of the system. By means of this variable, the complex dynamics of the experimental system could be analyzed simplifying the analysis of the fluid dynamics. The system of equations used here has been studied numerically by different authors [2, 10].

We can see the stationary states obtained when the control parameter (temperature) is increased and it is possible to relate it to the bifurcations in the system of equations. It can be observed how the system, firstly in the zero double point, breaks the spatial symmetry going then to one of the two possibilities that keep a subset of the symmetry (the diagonal). Each possibility of symmetry represent a spatial attractor.

After the spatial break of symmetry the sequence of bifurcations follows with a time-dependent regime. A further increase in the control parameter brings the system throughout a Hopf bifurcation to a limit cycle and then, to a chaotic attractor in presence of the symmetries of the square partially broken. A further increase in the control parameter brings the symmetric attractors to collide with the (0,0) point, opening the possibility to an hete-

roclinic conection. This heteroclinic connection produces a chaotic oscillation with a very rich dynamics between both attractors, each of them representing one of the preserved symmetries in the system.

In this work we present a detailed numerical analysis of the system of equations, identifying the influence of the different parameters on the dynamics. The aim is to detect the influence of those that could be used to control or synchronize the system in the chaotic attractor. After it, we present some results of simulations obtained when two identical experiments of this type are synchronized. To detect the different synchronization regimes we used the Lyapunov exponents calculated by the Runge-Kutta method for each variable. The system is hyperchaotic having more than one Lyapunov exponent positive. When two identical systems are coupled, we obtain a system of 8 dimensions instead of two of 4 and the space of phases can change as a function of the coupling parameters values.

As an interesting result we obtained generalized synchronization in windows as it has recently been shown in reference [11]. Moreover in this work the chaotic behavior is not suppressed. When synchronization takes place the system remains chaotic, but the order of chaos is lowered. This paper describes first the different dynamic states of the system and then the results on synchronization of two identical systems with symmetric coupling.

2. Bifurcations in systems with square symmetry

One of the simplest models to represent a real experiment in square symmetry can be found in [4]. However, the set of equations used in this work is over-simplified because only the symmetries that define the sub-space Z_2 are considered, and consequently, the experimental results can only be partially reproduced. In order to recover all the details of this experiment the same authors [5] introduce a system of equations that reproduce all the group of symmetries D_4 , the symmetries of the square. The bifurcation problem is now D_4 equivariant as long as non additional symmetries are generated by the boundary conditions of the box (hidden symmetries). D_4 represent the group of reflections (m) and rotations (ρ) of the square and the following equations system include all the elements of symmetry. Hidden symmetries have been used in [8] to reproduce the dynamics.

In our simulations we use the system of equations introduced in [2]:

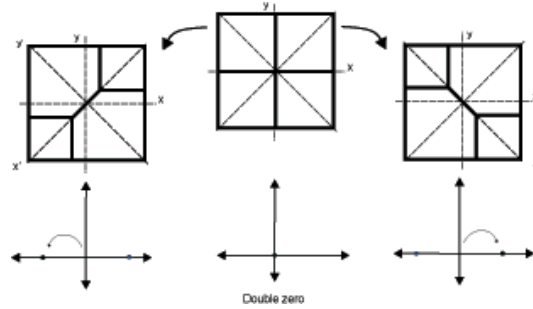


Figure 1: Spatial Bifurcations in the experiment and in the model. Symmetry breaks preserving the diagonal elements.

$$x' = y + \varepsilon^2 f z (zy - wx) \quad (1)$$

$$y' = \mu x + x(a(x^2 + z^2) + bz^2) + \varepsilon(\nu y + y(c(x^2 + z^2) + ez^2 + dx(xy + zw))) + \varepsilon^2 f w (zy - wx) \quad (2)$$

$$z' = w - \varepsilon^2 f x (zy - wx) \quad (3)$$

$$w' = \mu z + z(a(x^2 + z^2) + bx^2) + \varepsilon(\nu w + w(c(x^2 + z^2) + ex^2 + dz(xy + zw))) - \varepsilon^2 f y (zy - wx) \quad (4)$$

where (x, y, z, w) are the variables and $(a, b, c, d, e, f, \mu, \nu, \varepsilon)$ are parameters that must be adjusted to fit the experiment. In order to solve this system a 4th order Runge-Kutta method has been used and the solutions obtained were controlled by comparing to well known situations obtained by other authors that use the same equations (when available). In order to check our simulations we reproduce here the results obtained in [4] for the sequence of bifurcations appearing when the control parameter is increased. It can be seen in Figure 3. When the control parameter is sufficiently high to have the heteroclinic connection, the system is equivalent to four coupled oscillators. The phase space of this oscillators seem two pair of eyes distributed in the planes (x, y) and (z, w) . Each pair of oscillators are winded by the heteroclinic connection trajectories. In Figure 4 (a), we present temporal signals obtained for each variable; and the plane of phases $y(t)$ vs. $x(t)$ and $w(t)$ vs. $z(t)$ are shown in Figure 4 (b). The similitude between both planes is easily seen.

It is important to remark that in the model ε is a critical parameter. If we look for solutions representing the experiment, we need the ε value near zero because the equations system becomes unstable [2]. We adjusted the parameters in the model in order to fit during a certain time the temporal series obtained from the experiment.

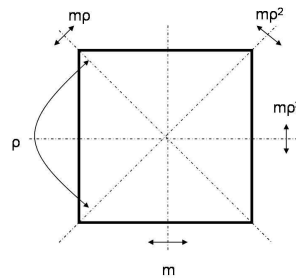


Figure 2: Generators of D_4 , the symmetry group of the square.

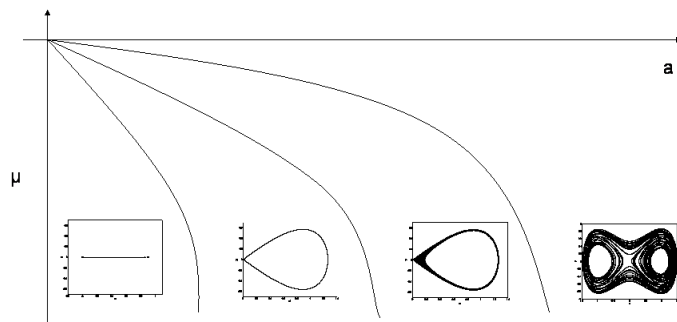


Figure 3: Bifurcations in the parameter space. The control parameter in the experiment is the heat flow, associated with μ .

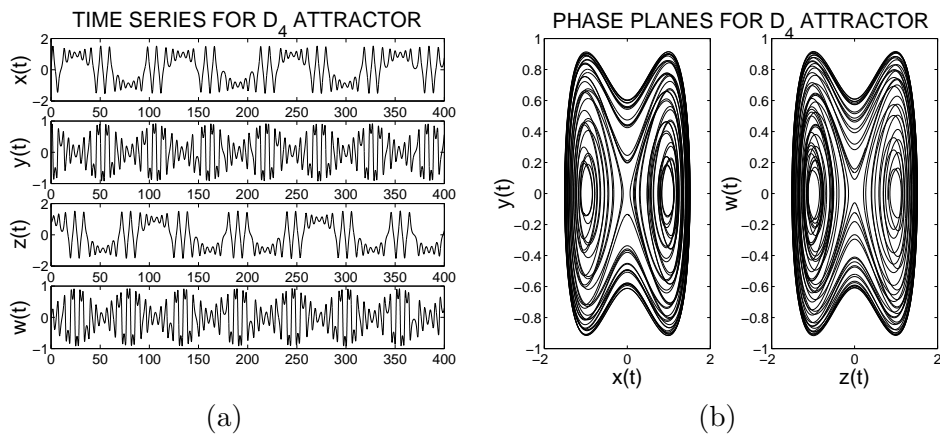


Figure 4: (a) shows the temporal data signals for each variable and attractors in the phase planes (x, y) and (z, w) are shown in (b).

3. Attempts to synchronize two identical systems

In order to analyze the synchronization possibilities two sets of equations like Eq. (1-4) has been coupled. We used a coupling scheme that can be symmetric or asymmetric and the coupling is a direct function of the error between both systems (acting as a feedback loop).

The equations for the coupled system can be seen in the set of Eq. (6-12) where variables are named by the subscripts (1,2) corresponding to each original system. The parameter θ change the strength of the coupling between the variables and $\theta \in [1, -1]$ controls the coupling symmetry. From the perfect symmetric coupling ($\theta = 0$), to the master-slave condition obtained when $\theta = (1, -1)$.

$$x'_1 = y_1 + \frac{\varepsilon_x}{2}(1 + \theta_x)(x_2 - x_1) \quad (5)$$

$$y'_1 = \mu x_1 + x_1(a(x_1^2 + z_1^2) + bz_1^2) + \frac{\varepsilon_y}{2}(1 + \theta_y)(y_2 - y_1) \quad (6)$$

$$z'_1 = w_1 + \frac{\varepsilon_z}{2}(1 + \theta_z)(z_2 - z_1) \quad (7)$$

$$w'_1 = \mu z_1 + z_1(a(x_1^2 + z_1^2) + bx_1^2) + \frac{\varepsilon_w}{2}(1 + \theta_w)(w_2 - w_1) \quad (8)$$

$$x'_2 = y_2 + \frac{\varepsilon_x}{2}(1 - \theta_x)(x_1 - x_2) \quad (9)$$

$$y'_2 = \mu x_2 + x_2(a(x_2^2 + z_2^2) + bz_2^2) + \frac{\varepsilon_y}{2}(1 - \theta_y)(y_1 - y_2) \quad (10)$$

$$z'_2 = w_2 + \frac{\varepsilon_z}{2}(1 - \theta_z)(z_1 - z_2) \quad (11)$$

$$w'_2 = \mu z_2 + z_2(a(x_2^2 + z_2^2) + bx_2^2) + \frac{\varepsilon_w}{2}(1 - \theta_w)(w_1 - w_2) \quad (12)$$

In order to detect synchronization windows we calculated the Lyapunov exponents when the systems are coupled on the variables x (with a coupling strength ε_x) and making $\varepsilon_y = \varepsilon_z = \varepsilon_w = 0$. To obtain a symmetric coupling the value of θ_x must be fixed to zero ($\theta_x = 0$). The system has now four positive Lyapunov exponents and the results against the coupling strength are shown in Figure 5.

In reference [11] the coupling of several chaotic 3-Dimensional systems have been analyzed (Rossler, Lorenz, etc.), and they found windows to synchronization observing the Lyapunov exponents behavior. Also in our system we found different windows where the coupled systems could be synchronized, but as a strong difference, here the chaos is not suppressed and variables are completely synchronized [13].

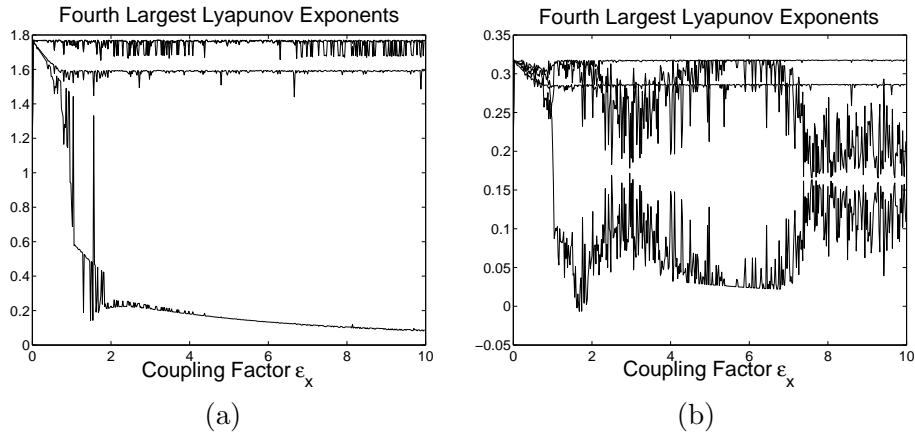


Figure 5: Lyapunov exponents for the coupled systems.

4. Conclusions

The Takens Bogdanov bifurcation equations can be used to analyze the space-time synchronization between two systems with square symmetry (like the experiment in reference [4]). Complete synchronization is achieved without chaos suppression. But some remarks about the relation between simulations and the experiment must be added. In the experiment represented by the equations considered here, the pattern obtained after the spatial bifurcations becomes time dependent. Under this conditions, the convective fluid layer receive a stationary flow of heat from a heater below and transfer it to the air in the upper side of the layer, but transformed into a time dependent heat flow. Physically considered this means that the system must store during a certain time a part of the total flow. Continuity requires to conserve the mean flow at the output equal to the stationary flow at the input. That is, the fluid layer need to transfer the heat flow modulating it in amplitude by the heat stored in the system. This produces, in consequence, a time dependent convective pattern following these modulations. In our experiment the flow mean value was modulated 10in amplitude (approx.) with quasiperiodic chaotic fluctuations. Our model focuses on the time dependent variable around the mean flow (the instantaneous value less the mean value). A scheme to clarify this can be seen in Figure 6. In the pattern, the projection of the diagonal length: ($x = d \cos \alpha$), follows the evolution of temperature in a wide range of the control parameter, as was demonstrated in [12]. The mean flow is a stationary quantity that can easily be calculated by the normal heat transfer equations.

Another mathematical restriction must be remarked. The system of equations considered here is very useful to represent the bifurcation process in a

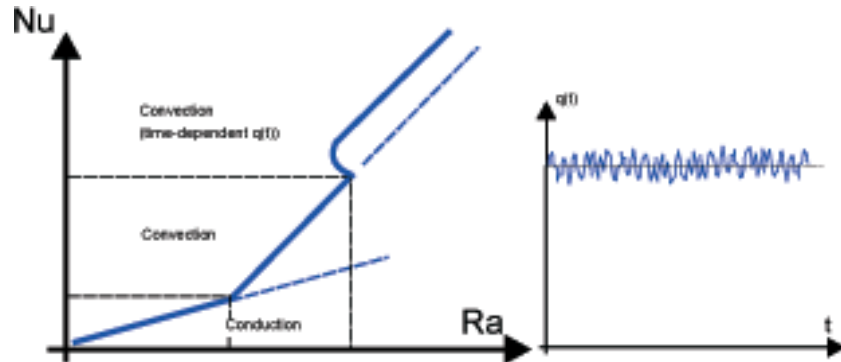


Figure 6: Scheme to clarify the time dependent flow magnitude against its mean value. Nu is the Nusselt number, a measure of the effective heat flow and Ra is the Rayleigh number, the non-dimensional control parameter [12].

space with symmetry D_4 . However it has some limitations if we need too long temporal series because the solution becomes unstable. In this case the model is not valid, unless we make $\varepsilon = f(x, y, z, w, t)$, which is not possible. As we have noted, variables (x, y, z, w) have a physical meaning, they define the size of the diagonals in the square, that is, a magnitude independent proportional to the instantaneous heat flow.

It is important to remember that the model is constructed not with the physical variables (velocity and temperature fields), but with a scalar projection of them defined on the pattern, that can be measured more easily. In short, the model represents the dynamic observed on the pattern and synchronization was achieved in the simulations, in the variables describing the movements of the pattern.

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