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CONTINUOUS CONVERGENCE AND DUALITY OF LIMITS OF TOPOLOGICAL ABELIAN GROUPS

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ABSTRACT. We find conditions under which direct and inverse limits of arbitrary indexed systems of topological Abelian groups are related via the duality defined by the continuous convergence structure. This generalizes known results by Kaplan about duality of direct and inverse sequences of locally compact Abelian groups.

1. Introduction

Given a topological Abelian group G, its group of continuous characters ΓG endowed with the compact open topology τ_{co} is another topological group, usually denoted by $G^{\wedge}$ and called the dual of G. The duality theorem of Pontryagin-van Kampen states that a locally compact Abelian (LCA) group G is topologically isomorphic to its bidual group $(G^{\wedge})^{\wedge}$ by means of the natural evaluation mapping. This theorem lies at the core of abstract harmonic analysis on locally compact Abelian groups and its extension to more general groups gives rise to the notion of reflexive group.

The original results of Pontryagin-van Kampen can be generalized to more general topological Abelian groups by means of two different duality theories. That is, given a topological Abelian group G we may consider ΓG endowed with either the compact open topology τ_{co}, obtaining $G^{\wedge}$ the Pontryagin dual (P-dual), or the continuous convergence structure Λ_c, obtaining a convergence group denoted by $\Gamma_c G$ that we call the c-dual of G. The convergence structure Λ_c has the advantage of making the evaluation mapping $\omega\colon \Gamma G \times G \to \mathbb{T}$ continuous although it is not usually topological. For a locally compact Abelian group G there is no difference between τ_{co} and Λ_c in ΓG. Hence the theorem of Pontryagin-van Kampen can be understood in the framework of the two dualities. There are many extensions of this theorem obtained for P-duality. We give as examples the ones by Kaplan [9], [10], Smith [15], Banaszczyk [2] or Pestov [14] among others. The approach of c-duality has also been fruitfully used in the works of Binz, Butzmann and others. The recent book of Beattie and Butzmann [3] provides an excellent overview of convergent structures and contains many relevant results in this direction.

A frequently used method to extend a property of a class of groups to a larger class is to take direct or inverse limits. There are situations where this method

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can be used to extend the known members of the class of reflexive groups. Kaplan proved that sequential direct and inverse limits of locally compact Abelian groups are P-reflexive and also that the P-dual of a sequential direct (inverse) limit is the inverse (direct) limit of the corresponding sequence of P-duals [10]. However, there is an old example due to Leptin [11] of an inverse limit of P-reflexive groups that is not P-reflexive.

The aim of the present article is to show that under some conditions, direct and inverse limits are related via c-duality. Working in the c-duality setting allows us to get rid of the requirement of countability of the index set that is present in Kaplan's results mentioned above. Countability is also needed in [1] where the authors prove that certain direct and inverse limits of sequences of P-reflexive Abelian groups that are metrizable or k_ω-spaces are P-reflexive and dual of each other. These results have been recently extended by Glöckner and Gramlich in [7].

We first study when the c-dual of a direct limit is the inverse limit of the c-dual system. Here, a crucial fact is that in the category of continuous convergence Abelian groups, the natural map η from a group to its c-bidual is continuous.

We then proceed to study under which conditions the c-dual of the limit of an inverse system is the direct limit of the c-dual system. This is a delicate problem that cannot be solved by categorical arguments only. The usual construction of the direct limit as a quotient group of the coproduct of the groups in the system gives a hint of where the difficulties come from. In P-duality the P-dual of the product is not always the coproduct.[1] This difficulty disappears in the framework of c-reflexivity [3]. However further work is needed to prove c-duality between general inverse and direct limits.

2. Convergence groups and c-duality

We introduce in this section the category of convergence Abelian groups denoted by CAG and the notion of c-duality. For an up to date introduction to convergence Abelian groups we refer the reader to the monograph [3].

First recall some basic notions about convergence spaces.

A *convergence structure* on a set X consists of a map $\lambda\colon X \to 2^{\mathbb{F}(X)}$ where $\mathbb{F}$ is the set of all filters on X, such that for all $x \in X$ we have

i) The filter generated by x belongs to $\lambda(x)$.
ii) For all filters $\mathcal{F}, G \in \lambda(x)$, the intersection $\mathcal{F} \cap \mathcal{G}$ belongs to $\lambda(x)$.
iii) If $\mathcal{F} \in \lambda(x)$, then $\mathcal{G} \in \lambda(x)$ for all filters $\mathcal{G}$ on X finer than $\mathcal{F}$.

A *convergence space* (X, Λ) is a set with a convergence structure. See ([3], pp. 2ff), for a more detailed exposition.

The notion of convergence space generalizes that of topological space. A topological space has a natural convergence structure, given by the convergent filters in the topology, which makes it a convergence space. Note that there are well known convergence structures, like the almost sure convergence in measure theory, that do not come from a topology on the supporting set.

[1]Nickolas proved that the P-dual of a product of LCA groups coincides with the coproduct of the P-duals if and only if the index set is countable [13].

Many topological notions that can be stated in terms of convergence of filters (such as continuity, open and closed sets, cluster point, compactness, etc) have their corresponding definitions for convergence spaces.

A *convergence group* is a group endowed with a convergence structure compatible with the group structure. Clearly every topological group is a convergence group and it can be treated in this way.

Let CAG be the category of convergence Abelian groups whose objects are convergence Abelian groups and whose morphisms are continuous homomorphisms. For two objects G and H in CAG, the group of morphisms from G to H will be denoted by CAG(G, H). The category TAG of topological Abelian groups and continuous homomorphisms is a full subcategory of CAG.

Consider the multiplicative group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with the Euclidean topology and denote by ΓG the group of morphisms CAG$(G, \mathbb{T})$.

We now define a convergence structure that makes ΓG a convergence group with nice properties. The *continuous convergence structure* Λ_c in ΓG is the coarsest convergence structure for which the evaluation mapping $\omega\colon \Gamma G \times G \to \mathbb{T}$ is continuous[2] ($\Gamma G \times G$ has the natural product convergence).

That is: A filter $\varPhi$ of ΓG converges continuously to ϕ if and only if $\omega(\varPhi \times \mathcal{F}) = \varPhi(\mathcal{F})$ converges to $\phi(x)$ in $\mathbb{T}$, for every $\mathcal{F} \to x$ in G. Here $\varPhi \times \mathcal{F}$ denotes the filter generated by the products $\Phi \times F$ and $\omega(\varPhi \times \mathcal{F}) = \varPhi(\mathcal{F})$ denotes the filter generated by the sets $\Phi(F)$, with $\Phi \in \varPhi$, $F \in \mathcal{F}$.

For any object G in CAG, we have that ΓG with the continuous convergence structure Λ_c is a Hausdorff convergence group ([3], 8.1) named the *convergence dual group* of G (c-dual for short) and denoted by $\Gamma_c G$. By Hausdorff we mean that any filter in $\Gamma_c G$ has at most one limit. From now on we will consider all of our groups in the subcategory of Hausdorff convergence Abelian groups HCAG.

For each $f \in$ HCAG(G, H), we can define the adjoint homomorphism $\Gamma_c f \in$ HCAG$(\Gamma_c H, \Gamma_c G)$ by $\Gamma_c f(\chi) = \chi \circ f$ for $\chi \in \Gamma_c H$. Thus $\Gamma_c(-)$ is a contravariant functor from HCAG to HCAG (or a covariant functor from HCAG^{op} to HCAG). There is a natural transformation κ from the identity functor in HCAG to the covariant functor $\Gamma_c\Gamma_c(-) := \Gamma_c(\Gamma_c(-))$. This can be described by $\kappa_G\colon G \to \Gamma_c\Gamma_c G$ where $\left[\kappa_G(x)\right](\chi) = \chi(x)$ for any $x \in G$ and $\chi \in \Gamma_c G$. Note that if the starting group G is a topological group, then the continuous convergence in its c-bidual $\Gamma_c\Gamma_c G$ is also topological (see [6]). A convergence Abelian group G is said to be *c-reflexive* if κ_G is an isomorphism in HCAG. The continuity of $\omega\colon \Gamma_c G \times G \to \mathbb{T}$ implies that κ_G is also continuous and hence a morphism in HCAG$(G, \Gamma_c\Gamma_c G)$.

We now relate c-reflexivity to the classical Pontryagin reflexivity. Recall that for a group G in HTAG, ΓG with the compact open topology τ_{co} is a topological group usually denoted by $G^{\wedge}$. The group G is called *Pontryagin-reflexive* or *P-reflexive*, if the evaluation $\sigma_G \to G^{\wedge\wedge}$ is a topological isomorphism. Note that this evaluation may not even be a morphism in HTAG, since it may not be continuous. The duality theorem of Pontryagin-van Kampen was originally stated for groups in LCA. For a group G in this category, τ_{co} and Λ_c coincide in

[2]Note that in the Pontryagin setting the continuity of $\omega\colon G^{\wedge} \times G \to \mathbb{T}$ is a strong requirement since it forces any reflexive group G to be locally compact [12].

ΓG, hence in LCA there are no differences between P-duality and c-duality.[3] Therefore the original results of Pontryagin-van Kampen can be generalized in two directions. Given a group G, consider in ΓG either the compact open topology to study P-reflexivity (as in Pontryagin duality theory), or the continuous convergence structure to study c-reflexivity. We will adopt the latter point of view in the remaining sections.

3. Direct and inverse limits of convergence groups

A directed set $\mathcal{A}$ can be considered as a category where the objects are the elements $\alpha \in \mathcal{A}$ and the set of morphisms $\mathcal{A}(\alpha, \beta)$ consists of only one element if $\alpha \leq \beta$ and is empty otherwise. A *direct system* in HCAG is a covariant functor D from a directed set $\mathcal{A}$ to HCAG. We use the notation $\{G_\alpha, f_\alpha^\beta, \mathcal{A}\}$ for a direct system, where $G_\alpha = D(\alpha)$ are the groups and $f_\alpha^\beta = D\left(\mathcal{A}(\alpha, \beta)\right)$ the linking maps.

A *direct limit* or *inductive limit* for a direct system $\{G_\alpha, f_\alpha^\beta, \mathcal{A}\}$ in HCAG is a pair $(\varinjlim G_\alpha, \{p_\alpha\}_{\alpha\in\mathcal{A}})$, where $\varinjlim G_\alpha$ is an object in HCAG and the p_α's are morphisms in HCAG$(G_\alpha, \varinjlim G_\alpha)$ such that $p_\alpha = p_\beta \circ f_\alpha^\beta$ for $\alpha \leq \beta$, satisfying the following universal property: Given an object G' in HCAG and morphisms p'_α in HCAG(G_α, G') for all $\alpha \in \mathcal{A}$ such that $p'_\alpha = p'_\beta \circ f_\alpha^\beta$ whenever $\alpha \leq \beta$, there is a unique morphism p in HCAG$(\varinjlim G_\alpha, G')$ such that $p'_\alpha = p \circ p_\alpha$.

Dually, an inverse system in HCAG is a contravariant functor I from $\mathcal{A}$ to HCAG (or equivalently a covariant functor from $\mathcal{A}$ to HCAG^{op}, the opposite category). We will denote a generic inverse system by $\{G_\alpha, g_\beta^\alpha, \mathcal{A}\}$ and an *inverse limit* or *projective limit* by a pair $(\varprojlim G_\alpha, \{\pi_\alpha\}_{\alpha\in\mathcal{A}})$, where $\pi_\alpha\colon \varprojlim G_\alpha \to G_\alpha$.

In order to describe the standard constructions of inverse and direct limits in HCAG we first recall the notions of products and coproducts in this category.

Let $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ be a family in HCAG and let $\prod G_\alpha$ be the (algebraic) product. The *product convergence structure* on the group $\prod G_\alpha$ is the initial convergence structure with respect to the projections $\pi_\alpha\colon \prod G_\alpha \to G_\alpha$. This convergence structure makes $\prod G_\alpha$ an object in HCAG.

A filter $\mathcal{F}$ converges to an element $x \in \prod G_\alpha$ if and only if, for each $\alpha \in \mathcal{A}$, $\pi_\alpha(\mathcal{F})$ converges to $\pi_\alpha(x)$ in G_α. Observe that if all the convergence groups of the family $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ are topological, then its convergence product is also topological.

The inverse limit of an inverse system $\{G_\alpha, g_\beta^\alpha, \mathcal{A}\}$ in HCAG, can be constructed as the following subgroup of the product $\prod G_\alpha$,

$$\left\{(x_\alpha)_{\alpha\in\mathcal{A}} \in \prod G_\alpha : g_\beta^\alpha(x_\beta) = x_\alpha\right\} .$$

The algebraic coproduct of Abelian groups $\bigoplus_{\alpha\in\mathcal{A}} G_\alpha$ is the group of all $x \in \prod G_\alpha$ such that $\{\alpha \in \mathcal{A}\colon\ \pi_\alpha(x) \neq e_{G_\alpha}\}$ is finite. The *coproduct convergence structure* is defined as the finest group convergence structure making the inclusions $i_\alpha\colon G_\alpha \to \bigoplus G_\alpha$ continuous.

The group $\bigoplus G_\alpha$ with the coproduct convergence structure is an object of HCAG called the *coproduct convergence group* of the family $\{G_\alpha\}_{\alpha\in\mathcal{A}}$.

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[3] A metrizable topological Abelian group is P-reflexive if and only if it is c-reflexive [5]. However this equivalence is not true in general [6].

Considering the coproduct convergence on $\bigoplus G_\alpha$, the standard construction of the inductive limit in HCAG for a direct system $\{G_\alpha, f_\alpha^\beta, \mathcal{A}\}$ is the following

$$\varinjlim G_\alpha \cong (\bigoplus G_\alpha)/\bar{H},$$

where H is the subgroup generated by $\{i_\beta \circ f_\alpha^\beta(g_\alpha) - i_\alpha(g_\alpha)\colon\ \alpha \leq \beta;\ g_\alpha \in G_\alpha\}$, and $\bar{H}$ is the intersection of all the closed subgroups of G containing H.

4. Duality properties of limits

There are many interesting results published in the literature about c-duality of convergence groups. We will use two of them due to Beattie and Butzmann as the starting point of our study. The first result establishes the isomorphisms $\Gamma_c\left(\prod G_\alpha\right) \cong \bigoplus \Gamma_c G_\alpha$ and $\Gamma_c\left(\bigoplus G_\alpha\right) \cong \prod \Gamma_c G_\alpha$ where $(G_\alpha)_{\alpha\in\mathcal{A}}$, is any family of convergence Abelian groups. Consequently if the convergence groups (G_α) are all c-reflexive, both $\bigoplus G_\alpha$ and $\prod G_\alpha$ are also c-reflexive (pp. 214-215 of [3]).

Remark. Observe that if we work with arbitrary index sets we cannot translate this statement completely to the Pontryagin setting. The product of an arbitrary family of P-reflexive groups is P-reflexive, however the P-dual of the product cannot always be described as the coproduct of the P-dual system, as we noticed in the introduction.

The second result by Beattie and Butzmann (p. 229 of [3]) shows that the limit of an inverse system of locally compact topological groups is c-reflexive. We have further explored the duality relation between direct and inverse limits. Our first result describes the c-dual of the direct limit and it follows directly from categorical arguments.

THEOREM (4.1). *Let $\{G_\alpha, f_\alpha^\beta, \mathcal{A}\}$ be a a direct system of convergence groups. Then*

$$\Gamma_c(\varinjlim G_\alpha) \cong \varprojlim \Gamma_c G_\alpha$$

Proof. For each pair G and H of objects in HCAG and morphism $f\colon G \to \Gamma_c H$, there is a unique morphism $f'\colon H \to \Gamma_c G$ such that $\Gamma_c(f')\circ\kappa_G = f$. In fact, for $h \in H$ and $g \in G$, $f'(h)(g) = f(g)(h)$ and the map $A\colon$ HCAG$(G, \Gamma_c H) \to$ HCAG$(H, \Gamma_c G)$ which maps f to f' is continuous. Hence, the functor $\Gamma_c(-)\colon$ HCAG$^{op} \to$ HCAG is right adjoint to $\Gamma_c(-)\colon$ HCAG $\to$ HCAG^{op} and consequently, the contravariant functor $\Gamma_c(-)\colon$ HCAG $\to$ HCAG transforms direct into inverse limits whenever they exist ([8], p. 307). Hence

$$\Gamma_c(\varinjlim G_\alpha) \cong \varprojlim \Gamma_c G_\alpha \qquad \square$$

The c-dual of the inverse limit cannot be obtained in such a natural way and requires restrictions on the groups and morphisms, which we proceed to describe.

Denote $\mathbb{T}_+ = \{z \in \mathbb{T}\,|\operatorname{Re} z \geq 0\}$. For a convergence group G, the *polar* of a subset $A \subset G$ is the set $A^\rhd = \{\chi \in \Gamma G\colon \chi(A) \subset \mathbb{T}_+\}$ and the inverse polar of a subset $B \subset \Gamma G$ is $B^\lhd = \{x \in G\colon \chi(x) \subset \mathbb{T}_+ \text{ for all } \chi \in B\}$.

Let G be an object of HCAG. A subgroup H of G is called *dually closed* in G if for every $x \in G \setminus H$ there exists a character $\chi \in \Gamma G$ with $\chi(H) = e_\mathbb{T}$ and

$\chi(x) \neq e_{\mathbb{T}}$. A subgroup H of G is called *dually embedded* if every character of H extends to a character of G. Note that a subgroup H of G is dually closed in G if and only if $H = H^{\triangleright\triangleleft}$.

PROPOSITION (4.2). (1) *Let $\{G_\alpha, f_\alpha^\beta, \mathcal{A}\}$ be a direct system of convergence groups and $H = gp\{i_\alpha(x_\alpha) - i_\beta \circ f_\alpha^\beta(x_\alpha)\colon \alpha \leq \beta;\ x_\alpha \in G_\alpha\}$. Then*

$$H^{\triangleright} = \varprojlim \Gamma_c G_\alpha.$$

(2) *Let $\{G_\alpha, g_\beta^\alpha, \mathcal{A}\}$ be an inverse system of convergence groups where the limit maps π_α have dense images. Let $L = gp\{i_\alpha(\varphi_\alpha) - i_\beta \circ \Gamma_c(g_\beta^\alpha)(\varphi_\alpha) : \alpha \leq \beta, \varphi_\alpha \in \Gamma_c G_\alpha\}$. Then*

$$(\varprojlim G_\alpha)^{\triangleright} = L.$$

Proof. First part:

Given $(\varphi_\alpha)_{\alpha\in\mathcal{A}} \in \prod \Gamma_c G_\alpha$ and $x_\alpha \in G_\alpha$, the following equalities hold:

$$(\varphi_\alpha)(i_\alpha(x_\alpha) - i_\beta \circ f_\alpha^\beta(x_\alpha)) = \varphi_\alpha(x_\alpha) - \varphi_\beta(f_\alpha^\beta(x_\alpha)) = \varphi_\alpha(x_\alpha) - \Gamma_c f_\alpha^\beta(\varphi_\beta)(x_\alpha)\,.$$

From here it follows, on the one hand, that if $(\varphi_\alpha)_{\alpha\in\mathcal{A}} \in \varprojlim \Gamma_c G_\alpha$, then $(\varphi_\alpha)(i_\alpha(x_\alpha) - i_\beta \circ f_\alpha^\beta(x_\alpha)) = e_{\mathbb{T}}$ and on the other hand if $(\varphi_\alpha)_{\alpha\in\mathcal{A}} \in H^{\triangleright}$, then $\Gamma_c f_\alpha^\beta(\varphi_\beta) = \varphi_\alpha$ since $(\Gamma_c f_\alpha^\beta(\varphi_\beta) - \varphi_\alpha)(x_\alpha) = e_{\mathbb{T}}$ for all $x_\alpha \in G_\alpha$.

Second part:

If $(x_\alpha)_{\alpha\in\mathcal{A}} \in \varprojlim G_\alpha$, we have that $g_\beta^\alpha(x_\beta) = x_\alpha$, hence

$$\begin{aligned}(i_\alpha(\varphi_\alpha) - i_\beta \circ \Gamma_c(g_\beta^\alpha)(\varphi_\alpha))(x_\alpha)_{\alpha\in\mathcal{A}} &= \varphi_\alpha(x_\alpha) - \left(\Gamma_c(g_\beta^\alpha)(\varphi_\alpha)\right)(x_\beta)\\ &= \varphi_\alpha(x_\alpha) - \varphi_\alpha\left(g_\beta^\alpha(x_\beta)\right)\\ &= \varphi_\alpha(x_\alpha) - \varphi_\alpha(x_\alpha) = e_{\mathbb{T}},\end{aligned}$$

and we have proven that $L \subset (\varprojlim G_\alpha)^{\triangleright}$.

We are left to prove the opposite inclusion. Any element $(\varphi_\alpha)_{\alpha\in\mathcal{A}} \in (\varprojlim G_\alpha)^{\triangleright}$ can be represented as a finite sum

$$(\varphi_\alpha)_{\alpha\in\mathcal{A}} = i_{\alpha_1}(\varphi_{\alpha_1}) + \cdots + i_{\alpha_k}(\varphi_{\alpha_k})\,.$$

where $\alpha_k \geq \alpha_1, \ldots, \alpha_{k-1}$

Consider now an arbitrary element $x_{\alpha_k} \in \pi_{\alpha_k}(\varprojlim G_\alpha)$ and let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be an element of the inverse limit with α_k coordinate x_{α_k}. We know that $g_\beta^\alpha(x_\beta) = x_\alpha$, $\alpha \leq \beta$ and since $(\varphi_\alpha)_{\alpha\in\mathcal{A}}$ is in the polar of $\varprojlim G_\alpha$, we have

$$\begin{aligned}&((\Gamma_c(g_{\alpha_k}^{\alpha_1}))(\varphi_{\alpha_1}) + \cdots + (\Gamma_c(g_{\alpha_k}^{\alpha_{k-1}}))(\varphi_{\alpha_{k-1}}) + \varphi_{\alpha_k})(x_{\alpha_k})\\ &\qquad= (\varphi_{\alpha_1} g_{\alpha_k}^{\alpha_1} + \cdots + \varphi_{\alpha_{k-1}} g_{\alpha_k}^{\alpha_{k-1}} + \varphi_{\alpha_k})(x_{\alpha_k})\\ &\qquad= \varphi_{\alpha_1}(x_{\alpha_1}) + \cdots + \varphi_{\alpha_k}(x_{\alpha_k})\\ &\qquad= (\varphi_\alpha)_{\alpha\in\mathcal{A}}((x_\alpha)_{\alpha\in\mathcal{A}}) = e_{\mathbb{T}}\end{aligned}$$

and hence, since $\pi_{\alpha_k}(\varprojlim G_\alpha)$ is dense in G_{α_k},

$$\left((\Gamma_c(g_{\alpha_k}^{\alpha_1}))(\varphi_{\alpha_1}) + \cdots + (\Gamma_c(g_{\alpha_k}^{\alpha_{k-1}}))(\varphi_{\alpha_{k-1}}) + \varphi_{\alpha_k}\right) = e_{\Gamma_c G_{\alpha_k}}\,.$$

We can now subtract this term from the expression for $(\varphi_\alpha)_{\alpha\in\mathcal{A}}$ which is enough to obtain our result. More concretely,

$$\begin{aligned}(\varphi_\alpha)_{\alpha\in\mathcal{A}} &= i_{\alpha_1}(\varphi_{\alpha_1}) + \cdots + i_{\alpha_k}(\varphi_{\alpha_k})\\ &= i_{\alpha_1}(\varphi_{\alpha_1}) + \cdots + i_{\alpha_k}(\varphi_{\alpha_k})\\ &\quad - i_{\alpha_k}\Big((\Gamma_c(g^{\alpha_1}_{\alpha_k}))(\varphi_{\alpha_1}) + \cdots + (\Gamma_c(g^{\alpha_{k-1}}_{\alpha_k}))(\varphi_{\alpha_{k-1}}) + \varphi_{\alpha_k}\Big)\\ &= i_{\alpha_1}(\varphi_{\alpha_1}) - i_{\alpha_k}(\Gamma_c(g^{\alpha_1}_{\alpha_k}))(\varphi_{\alpha_1}) + \dots\\ &\quad + i_{\alpha_{k-1}}(\varphi_{\alpha_{k-1}}) - i_{\alpha_k}(\Gamma_c(g^{\alpha_{k-1}}_{\alpha_k}))(\varphi_{\alpha_{k-1}}) + i_{\alpha_k}(\varphi_{\alpha_k}) - i_{\alpha_k}(\varphi_{\alpha_k}),\end{aligned}$$

from which we conclude $(\varprojlim G_\alpha)^\triangleright \subset L$. □

We describe the c-dual of the inverse limits in the class of *Nuclear groups*. Roughly speaking a Hausdorff Abelian group G is Nuclear if each neighborhood of zero contains another neighborhood which is "sufficiently small"[4]. This class of groups, introduced by Banaszczyk in [2], has good permanence properties — subgroups, quotients and products of nuclear groups are nuclear groups. Locally compact groups are nuclear and the groups underlying nuclear locally convex topological vector spaces are also in the class of nuclear groups. Banaszczyk succeeded in generalizing many properties of LCA groups to nuclear groups.

LEMMA (4.3). *Every subgroup H of a nuclear group G is dually embedded and $\Gamma_c i\colon \Gamma_c G \to \Gamma_c H$ is a quotient mapping with kernel $H^\triangleright$.*

Proof. See Corollary 8.3 in [2] and Corollary 8.4.10 in [3]. □

Our first description of the c-dual of an inverse limit also requires some restriction on the limit maps.

THEOREM (4.4). *Let $\{G_\alpha, g^\alpha_\beta; \mathcal{A}\}$ be an inverse system of nuclear groups where the limit maps π_α have dense images. Then*

$$\Gamma_c(\varprojlim G_\alpha) \cong \varinjlim \Gamma_c G_\alpha$$

Proof. We have by (4.2) (2) that

$$(\varprojlim G_\alpha)^\triangleright = gp\{i_\alpha(\varphi_\alpha) - i_\beta \circ \Gamma_c(g^\alpha_\beta)(\varphi_\alpha), : \alpha \le \beta, \varphi_\alpha \in \Gamma_c G_\alpha\}.$$

It follows that $\varinjlim \Gamma_c G_\alpha$ is the quotient convergence group $(\bigoplus \Gamma_c G_\alpha)/(\varprojlim G_\alpha)^\triangleright$. But this is an object in HCAG isomorphic to $\Gamma_c(\prod G_\alpha)/(\varprojlim G_\alpha)^\triangleright$. We still need to prove that $\Gamma_c(\varprojlim G_\alpha)$ is isomorphic to this object. In order to do that we use Lemma (4.3) about subgroups of nuclear groups:

Since all groups G_α are nuclear groups the product $\prod G_\alpha$ is nuclear, therefore by Lemma (4.3), $\Gamma i\colon \Gamma_c(\prod G_\alpha) \to \Gamma_c(\varprojlim G_\alpha)$ is a quotient mapping with kernel $(\varprojlim G_\alpha)^\triangleright$ which induces an isomorphism $\psi\colon \Gamma_c(\prod G_\alpha)/(\varprojlim G_\alpha)^\triangleright \to \Gamma_c(\varprojlim G_\alpha)$ in the category HCAG. Hence the assertion follows. □

[4] A Hausdorff Abelian group is called Nuclear if it satisfies the following condition: Given an arbitrary neighborhood U of e_G, $c > 0$ and $m = 1, 2, \dots$, there exists a vector space E and two pre-Hilbert seminorms p, q on E with $d_k(B_p, B_q) \le ck^{-m}$, where d_k is the kth Kolmogorov diameter and $k = 1, 2, \dots$, ([2] p. 72)

We now give an alternative description of the c-dual of an inverse limit without any condition on the limit maps. Let G be a convergence group. We will say that G *has enough characters* if $\kappa_G \colon G \to \Gamma_c \Gamma_c G$ is injective, i.e., if for all $x \in G$, $x \neq e_G$ there exists $\chi \in \Gamma_c G$ such that $\chi(x) \neq e_{\mathbb{T}}$. Given an arbitrary convergence group G, it is easy to check that $G/\ker(\kappa_G)$ is a convergence group with enough characters.

Denote by $\mathrm{HCAG}_{\kappa_{1:1}}$ the category of convergence groups with enough characters, we can define a full functor $F\colon$ HCAG $\to \mathrm{HCAG}_{\kappa_{1:1}}$ by $F(G) = G/\ker(\kappa_G)$. The functor F is left adjoint to the inclusion functor $\mathrm{HCAG}_{\kappa_{1:1}} \to$ HCAG and hence it preserve direct limits, i.e., $F(\varinjlim G_\alpha) = \varinjlim (FG_\alpha)$.

LEMMA (4.5). *Let G be a Hausdorff convergence group and H a closed subgroup of G, then $F(G/H) \cong G/H^{\triangleright\triangleleft}$.*

Proof. Since $F(G/H) \cong \frac{G/H}{\ker(\kappa_{G/H})}$, it is enough to see that $\ker(\kappa_{G/H})$ is precisely $H^{\triangleright\triangleleft}/H$. Now for $x \in G$, $\kappa_{G/H}[x] = e_{\Gamma_c\Gamma_c(G/H)}$ iff $\chi[x] = e_{\mathbb{T}}$ for all $\chi \in \Gamma_c(G/H)$ which is the same as the statement: $\widetilde{\chi}(x) = e_{\mathbb{T}}$ for all $\widetilde{\chi} \in \Gamma_c G$ such that $\widetilde{\chi}(H) = e_{\mathbb{T}}$ and this occurs if and only if $x \in H^{\triangleright\triangleleft}$. □

THEOREM (4.6). *Let $\{G_\alpha, g^\alpha_\beta, \mathcal{A}\}$ be an inverse system of complete nuclear topological groups. Then*

$$\Gamma_c(\varprojlim G_\alpha) \cong F(\varinjlim \Gamma_c G_\alpha)\,.$$

Proof. Note that a nuclear group is complete if and only if it is c-reflexive (see [4]). We know that $\varprojlim G_\alpha$ is a subgroup of $\prod G_\alpha$, which in turn is a nuclear group. Hence by 8.4.5 in [3] $\Gamma_c(i)\colon \Gamma_c(\prod G_\alpha) \to \Gamma_c(\varprojlim G_\alpha)$ is a quotient map with kernel $(\varprojlim G_\alpha)^{\triangleright}$. This map induces an isomorphism $\Gamma_c(\prod G_\alpha)/(\varprojlim G_\alpha)^{\triangleright} \to \Gamma_c(\varprojlim G_\alpha)$ in HCAG.

Denote by $L = gp\{i_\alpha(\varphi_\alpha) - i_\beta \circ \Gamma_c(g^\alpha_\beta)(\varphi_\alpha) : \alpha \leq \beta, \varphi_\alpha \in \Gamma_c G_\alpha\}$

Now by (4.2).1 we have that $L^{\triangleright} = \varprojlim(\Gamma_c\Gamma_c G_\alpha) \cong \varprojlim G_\alpha$. Hence $L^{\triangleright\triangleright} \cong (\varprojlim G_\alpha)^{\triangleright}$. The c-reflexivity of $\bigoplus \Gamma_c G_\alpha$ yields $(\varprojlim G_\alpha)^{\triangleright} = L^{\triangleright\triangleleft}$. Finally

$$\begin{aligned}\Gamma_c(\varprojlim G_\alpha) &\cong \frac{\Gamma_c(\prod G_\alpha)}{(\varprojlim G_\alpha)^{\triangleright}} \cong \frac{\bigoplus \Gamma_c G_\alpha}{(\varprojlim G_\alpha)^{\triangleright}} \\ &= \frac{\bigoplus \Gamma_c G_\alpha}{L^{\triangleright\triangleleft}} = F\left(\frac{\bigoplus \Gamma_c G_\alpha}{\overline{L}}\right) = F(\varinjlim \Gamma_c G_\alpha).\end{aligned}$$

□

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