On Schwartz groups

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Abstract

In this paper we introduce a notion of a Schwartz group, which turns out to be coherent with the well known concept of a Schwartz topological vector space. We establish several basic properties of Schwartz groups and show that free topological Abelian groups, as well as free locally convex spaces, over a hemicompact k–space are Schwartz groups. We also prove that every hemicompact k–space topological group, in particular the Pontryagin dual of a metrizable topological group, is a Schwartz group.

1 Introduction

The notions of Schwartz and nuclear locally convex spaces were introduced by A. Grothendieck in [14] and [13], respectively. An intensive study of these spaces was made in [14, 15, 17, 24, 18, 27], and many other papers. Some relevant problems in the theory of Schwartz spaces have been solved only recently, see in this connection [6, 7].

Many important spaces in Analysis and its applications are either nuclear or Schwartz. Spaces in both classes satisfy some properties of finite-dimensional spaces (e. g. their bounded subsets are precompact) which general Banach spaces do not have.

A group version of the concept of a nuclear space, introduced by W. Banaszczyk in [5], has been proved to be useful in Harmonic Analysis and topological group theory. In this paper we define a group counterpart of the Schwartz notion. Our definition uses only group-theoretic tools. For the underlying additive group of a topological vector space, our notion appears to be the usual notion of a Schwartz space.

The paper is organized as follows:

In Section 2 we recall some necessary concepts of the theory of topological Abelian groups.

We start Section 3 defining Schwartz groups. After that, we obtain permanence properties for this class regarding subgroups, Hausdorff quotients, products and local isomorphisms, and show that bounded subsets of locally

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quasi–convex Schwartz groups are precompact. We finish this section proving that every Schwartz group can be embedded in a product of metrizable Schwartz groups.

We establish in Section 4 that the underlying additive group of a topological vector space E is a Schwartz group according to our definition if and only if E is a Schwartz space. At this point a rather unexpected result is obtained: there is a metrizable Schwartz group without nontrivial continuous characters. Such a group is obtained as a quotient group of a metrizable locally convex Schwartz space. It is known, however ([5, 8.6, 7.4]) that quotient groups of metrizable nuclear locally convex spaces are nuclear groups and hence their continuous characters separate their points. The remaining part of Section 4 is devoted to prove that all nuclear groups are Schwartz groups.

In the last section we deal with free topological Abelian groups. Here our main result is proved: if a completely regular Hausdorff topological space X is a hemicompact k-space, then the free topological Abelian group $A(X)$ is a locally quasi-convex Schwartz group. Note that for an infinite compact space X the group $A(X)$ is never nuclear ([3]). This result has also consequences out of the free group framework. We prove in particular that every hemicompact k–space topological group is Schwartz, as well as the Pontryagin dual group of any arbitrary metrizable topological Abelian group.

2 First definitions and results

All groups under consideration will be Abelian topological groups. The set of neighborhoods of the neutral element in the group G will be denoted by $\mathcal{N}_0(G)$.

We write $\mathbb{T}_+ = \{t \in \mathbb{T} : \operatorname{Re} t \geq 0\}$ where $\mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}$ is the compact torus. For an Abelian topological group G, the group of all continuous homomorphisms $\chi : G \to \mathbb{T}$, usually called *continuous characters*, with pointwise multiplication and endowed with the compact–open topology, is a topological group, denoted by $G^\wedge$ and called *character group* of G.

For $U \subset G$, the set $U^\triangleright := \{\chi \in G^\wedge : \ \chi(U) \subseteq \mathbb{T}_+\}$ is named the *polar* of U. We will say that U is *quasi-convex* if $U = \bigcap\limits_{\chi \in U^\triangleright} \chi^{-1}(\mathbb{T}_+)$.

A topological Abelian group is called *locally quasi-convex* if it has a basis of neighborhoods of zero formed by quasi-convex sets. (See [5].)

For a subset U of an Abelian group G, such that $0 \in U$, and a natural number n, we set $U_{(n)} := \{x \in G : x \in U, \ 2x \in U, \ \ldots, \ nx \in U\}$, and $U_{(\infty)} := \bigcap_{n \in \mathbb{N}} U_{(n)}$. For $\mathbb{T}_+ \subseteq \mathbb{T}$, we have $(\mathbb{T}_+)_{(n)} = \{e^{2\pi i t} : \ |t| \leq 1/4n\}$ and $(\mathbb{T}_+)_{(\infty)} = \{1\}$.

We shall list some properties of these new settings:

Lemma 2.1. *Let G be an Abelian group, U a subset of G with $0 \in U$ and $n \in \mathbb{N}$.*

1. *For any Abelian group H and any group homomorphism $\varphi : G \to H$ we have $\varphi(U_{(n)}) \subseteq \varphi(U)_{(n)}$.*

2. *For any subgroup S of G we obtain: $(S \cap U)_{(n)} = S \cap U_{(n)}$.*

3. *If G is a topological Abelian group and U is quasi–convex, then*
$$U_{(n)} = \bigcap_{\chi \in U^{\triangleright}} \chi^{-1}((\mathbb{T}_+)_{(n)})$$
and in particular $U_{(n)} + \overset{n}{\dots} + U_{(n)} \subseteq U$.

If U is a quasi-convex subset of G, then 2.1.3. implies that the family $\{U_{(n)} : n \in \mathbb{N}\}$ is a basis of neighborhoods of zero for a locally quasi-convex group topology $\mathcal{T}_U$ on G. We will use the notation G_U for the Hausdorff group $G/U_{(\infty)}$ associated with $(G, \mathcal{T}_U)$ and φ_U for the canonical map from G to G_U.

If U and V are quasi-convex subsets of a topological group G, such that $V \subseteq U$ the linking homomorphism $\varphi_{VU} : (G_V, \mathcal{T}_V) \to (G_U, \mathcal{T}_U)$, $\varphi_V(x) \mapsto \varphi_U(x) \in G_U$ is well defined and continuous.

Lemma 2.2. *Let (U_n) be a sequence of quasi-convex neighborhoods of zero in G which satisfy $U_{n+1} + U_{n+1} \subseteq U_n$. Then $H := \bigcap_{n \in \mathbb{N}} U_n$ is a subgroup of G and the sets $(\pi(U_n))_{n \in \mathbb{N}}$, where $\pi : G \to G/H$ denotes the canonical projection, form a neighborhood basis of zero of a locally quasi-convex group topology in G/H.*

Proof. Since quasi-convex sets are symmetric, it is straightforward to show that H is a subgroup of G. Every character $\chi \in U_n^{\triangleright}$ satisfies $\chi(H) = \{1\}$ and hence, by the definition of quasi-convexity, $H + U_n = U_n$. This implies that the sets $\pi(U_n)$ are quasi-convex and hence the assertion follows. □

3 Schwartz groups

Definition 3.1. *Let G be a Hausdorff topological Abelian group. We say that G is a Schwartz group if for every neighborhood of zero U in G there exists another neighborhood of zero V in G and a sequence (F_n) of finite subsets* of *G such that*
$$V \subseteq F_n + U_{(n)} \quad \textit{for every} \quad n \in \mathbb{N}.$$

Example 3.2. *From the definition we obtain directly that locally precompact groups are Schwartz groups.*

Remark 3.3. *As we shall see later on (cf. Theorem 4.5), local quasi–convexity and the notion of a Schwartz group defined in 3.1 are independent. The concept of a Schwartz group, however, is most fruitful when restricted to the class of locally quasi–convex groups. It is easy to prove that a locally quasi–convex group G is a Schwartz group if and only if for every quasi–convex neighborhood of zero U in G there exists another quasi–convex neighborhood of zero $V \subseteq U$ such that the linking homomorphism φ_{VU} is precompact (i.e. $\varphi_{VU}(W)$ is precompact in G_U for some neighborhood W in G_V).*

Proposition 3.4. *Let G and H be topological Abelian groups. Suppose that H is a Schwartz group and there exist $U_0 \in \mathcal{N}_0(G)$ and a map $\varphi : U_0 \to H$ such that*

(a) φ is continuous.

(b) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for every $x,\ y \in U_0$ such that $x + y \in U_0$.

(c) For every $U \in \mathcal{N}_0(G)$ with $U \subset U_0$ there exists $V \in \mathcal{N}_0(H)$ with $\varphi^{-1}(V_{(n)}) \subset U_{(n)}$ for every $n \in \mathbb{N}$.

Then G is a Schwartz group.

Proof. Fix a symmetric $U \in \mathcal{N}_0(G)$ with $U + U \subset U_0$. Let $V \in \mathcal{N}_0(H)$ be such that $\varphi^{-1}(V_{(n)}) \subset U_{(n)}$ for every $n \in \mathbb{N}$. Finally let $V' \in \mathcal{N}_0(H)$ be such that $V' + V' \subset V$. Since H is Schwartz, there exist $W \in \mathcal{N}_0(H)$ (which we may choose symmetric and contained in V), and a sequence of finite subsets $\tilde{F}_n \subset H$, with $W \subset \tilde{F}_n + V'_{(n)}$ for every $n \in \mathbb{N}$. Put $\tilde{F}_n = \{\tilde{f}_{1,n}, \tilde{f}_{2,n}, \dots, \tilde{f}_{i_n,n}\}$. For each $n \in \mathbb{N}$, define the (possibly empty) set of indices

$$I_n = \{i \in \{1, 2, \dots, i_n\} : \varphi^{-1}((\tilde{f}_{i,n} + V'_{(n)}) \cap W) \neq \emptyset\}.$$

For each $i \in I_n$ choose $f_{i,n} \in \varphi^{-1}((\tilde{f}_{i,n} + V'_{(n)}) \cap W)$, and define $F_n = \{f_{i,n} : i \in I_n\}$.

Let us show that $\varphi^{-1}(W) \subset F_n + U_{(n)}$ for every $n \in \mathbb{N}$; since φ is continuous, this will imply that G is a Schwartz group. Fix any $x \in \varphi^{-1}(W)$. For every $n \in \mathbb{N}$, $x \in \varphi^{-1}(\tilde{F}_n + V'_{(n)})$, hence there exists $j_n \in \{1, 2, \dots, i_n\}$ with $x \in \varphi^{-1}((\tilde{f}_{j_n,n} + V'_{(n)}) \cap W)$. Clearly $j_n \in I_n$, so we have chosen an $f_{j_n,n} \in \varphi^{-1}((\tilde{f}_{j_n,n} + V'_{(n)}) \cap W)$ above.

Now both x and $f_{j_n,n}$ are elements of $\varphi^{-1}(W) \subset \varphi^{-1}(V) \subset U$ and since $U + U \subset U_0$ it follows from (b) that

$$\varphi(x - f_{j_n,n}) = \varphi(x) - \varphi(f_{j_n,n}) = \varphi(x) - \tilde{f}_{j_n,n} + \tilde{f}_{j_n,n} - \varphi(f_{j_n,n})$$

which belongs to $V'_{(n)} + V'_{(n)} \subset V_{(n)}$. We deduce $x - f_{j_n,n} \in \varphi^{-1}(V_{(n)}) \subset U_{(n)}$ for every $n \in \mathbb{N}$ and the proof is complete. □

Remark 3.5. *From the proof of Prop. 3.4 it follows that if G is a Schwartz group, then the finite sets F_n and the neighborhood of zero V taking part in the definition can be chosen in such a way that $F_n \subseteq V$ for every $n \in \mathbb{N}$.*

Proposition 3.6. *(a) The class of Schwartz groups is a Hausdorff variety of topological Abelian group, i. e. the following properties hold:*

- *(a.1) Every subgroup of a Schwartz group is a Schwartz group.*
- *(a.2) The Cartesian product of an arbitrary family of Schwartz groups, equipped with the Tychonoff topology, is a Schwartz group.*
- *(a.3) Every Hausdorff quotient of a Schwartz group is a Schwartz group.*

(b) Let G and H be locally isomorphic topological Abelian groups. Then G is a Schwartz group if and only if H is a Schwartz group.

Proof. The proof of (a.2) is straightforward, (a.3) is an easy consequence of Lemma 2.1.1. and (a.1) and (b) are corollaries of Prop. 3.4. □

In our next result we will use the following notion of boundedness ([16]):

Definition 3.7. *Let G be a topological Abelian group and B a subset of G; B is said to be bounded if for every zero neighborhood U there exists a finite set $F \subseteq G$ and some $n \in \mathbb{N}$ such that $B \subseteq F + U + \overset{n}{\cdots} + U$.*

Clearly every precompact set is bounded according to this definition. Note also that in a locally convex vector space, the bounded sets are exactly those which are absorbed by any neighborhood of zero.

Proposition 3.8. *Let G be a locally quasi-convex Schwartz group and let B be a bounded subset of G. Then B is precompact.*

Proof. For a quasi-convex $U \in \mathcal{N}_0(G)$ there exist $V \in \mathcal{N}_0(G)$ and a sequence (F_n) of finite subsets of G such that $V \subseteq F_n + U_{(n)}$ for every $n \in \mathbb{N}$. On the other hand, by the boundedness of B there exist a finite set $F_0 \subseteq G$ and $m \in \mathbb{N}$ such that $B \subseteq F_0 + V + \overset{m}{\cdots} + V$. Hence, by 2.1.3,

$$\begin{aligned} B &\subseteq F_0 + V + \overset{m}{\cdots} + V \\ &\subseteq F_0 + (F_m + U_{(m)}) + \overset{m}{\cdots} + (F_m + U_{(m)}) \\ &\subseteq F_0 + (F_m + \overset{m}{\cdots} + F_m) + (U_{(m)} + \overset{m}{\cdots} + U_{(m)}) \\ &\subseteq F_0 + (F_m + \overset{m}{\cdots} + F_m) + U. \end{aligned}$$

□

The same property was obtained by J. Galindo ([12]) for bounded subsets of nuclear groups. In the next section we will see that every nuclear group is a Schwartz group.

Theorem 3.9. *Every (locally quasi–convex) Schwartz group can be embedded into a product of metrizable (locally quasi–convex) Schwartz groups.*

Proof. Let $\mathcal{U}_0$ be a neighborhood basis of G. Fix an arbitrary $U \in \mathcal{U}_0$. There exists a neighborhood U_1 and a sequence $(F_{1,n})_{n\in\mathbb{N}}$ of finite subsets of G such that $U_1 \subseteq F_{1,n} + U_{(n)}$ for every $n \in \mathbb{N}$. We may assume that U_1 is symmetric and satisfies $U_1 + U_1 \subseteq U$. Let us put $U_0 := U$ and suppose now that symmetric neighborhoods $U_1, \ldots, U_k$ and finite sets $F_{j,n}$ ($n \in \mathbb{N},\ j \in \{1, \ldots, k\}$) have been constructed which satisfy the following properties: $U_j + U_j \subseteq U_{j-1}$ and $U_j \subseteq F_{j,n} + (U_{j-1})_{(n)}$ for every $j \in \{1, \ldots, k\}$ and $n \in \mathbb{N}$. By the definition of a Schwartz group, it is possible to continue the construction.

The sequence $(U_0, U_1, U_2, \ldots)$ forms a neighborhood basis for a (not necessarily Hausdorff) group topology on G. The intersection $H_U := \bigcap_{n\in\mathbb{N}} U_n$ is a subgroup of G. Call p_U the canonical projection of G onto G/H_U. The sets $(p_U(U_n))_{n\in\mathbb{N}}$ generate a metrizable group topology on G/H_U. By 2.1.1. we have

$$p_U(U_j) \subseteq p_U(F_{j,n}) + p_U(U_{j-1})_{(n)}$$

for every j, $n \in \mathbb{N}$, and thus G/H_U is a metrizable (Hausdorff) Schwartz group.

With every $U \in \mathcal{U}_0$, we associate in the same way a metrizable Schwartz group topology on the corresponding quotient G/H_U, and consider the mapping

$$\Phi : G \to \prod_{U\in\mathcal{U}_0} G/H_U, \quad x \mapsto (p_U(x))_{U\in\mathcal{U}_0}.$$

It is clear that Φ is a continuous monomorphism. For $U \in \mathcal{U}_0$, we obtain, since $U_1 + H_U \subset U$,

$$\Phi(U) \supset \operatorname{im}\Phi \cap \left(p_U(U_1) \times \prod_{U'\in\mathcal{U}_0\setminus\{U\}} G/H_{U'} \right),$$

which shows that Φ is an embedding.

Now suppose that G is locally quasi–convex. The neighborhood basis $\mathcal{U}_0$, as well as the sequence $(U_0, U_1, U_2, \ldots)$ for each $U \in \mathcal{U}_0$, can be chosen to contain only quasi–convex sets. According to 2.2, the quotient groups G/H_U are locally quasi–convex as well. $\square$

4 Schwartz groups, Schwartz locally convex vector spaces and nuclear groups

In this section we will prove that both Schwartz spaces and nuclear groups are Schwartz groups.

Definition 4.1. *A topological vector space E is a Schwartz space if for every neighborhood of zero U in E, there exists another neighborhood V such that for every $\alpha > 0$ the set V can be covered by finitely many translates of αU.*

As in [26] or [21], local convexity is not required in our definition of Schwartz spaces (in [26, Example 6.4.4] an example of a non locally convex Schwartz F-space is given). Since the neighborhoods of zero in a topological vector space may be assumed to be radial, we can replace $\alpha > 0$ by $\frac{1}{n}$ (where $n \in \mathbb{N}$) in Definition 4.1. This easily leads to the following result:

Proposition 4.2. *Let E be a topological vector space. The following are equivalent:*

(a) E is a Schwartz space.

(b) The additive topological Abelian group underlying E is a Schwartz group.

A *locally convex vector group* is a real vector space endowed with a Hausdorff group topology which has a basis of neighborhoods of zero consisting of absolutely convex sets. This notion was defined by Raĭkov in [25].

Every absolutely convex and closed subset U of a locally convex vector group E, is quasi–convex (see the proof of Lemma (2.4) in [5]), and $E_{(U)} = \mathrm{sp}(U)/U_{(\infty)}$, endowed with the topology induced by $\mathcal{T}_U$, is a normed space.

Nuclear groups, as well as nuclear vector groups were introduced by Banaszczyk in [5]. The class of nuclear groups contains all locally compact Abelian groups and additive groups underlying nuclear vector groups. Moreover, it is closed with respect to the operations of taking subgroups and forming Hausdorff quotient groups, arbitrary products and countable direct sums. An intensive study of nuclear groups has been developed since their introduction, which has given rise to several important results (see [2] for a survey).

In [10, §8], it is proved that a nuclear vector group is a locally convex vector group E such that for every absolutely convex closed neighborhood of zero U in E there exists another absolutely closed convex neighborhood of zero $V \subseteq U$ such that $\widetilde{\varphi_{VU}} : \widetilde{E_{(V)}} \to \widetilde{E_{(U)}}$ is a nuclear operator where $\widetilde{X}$ stands for the completion of the normed space X, and $\widetilde{\varphi_{VU}}$ for the continuous extension of the canonical map φ_{VU} to the completions of both spaces $E_{(V)}$ and $E_{(U)}$.

Theorem 4.3. *Every nuclear group G is a locally quasi–convex Schwartz group.*

Proof. According to [5, 8.5], every nuclear group is locally quasi–convex. An important structural result ([5, 9.6]) implies that G is topologically isomorphic to H/K where H is a subgroup of a nuclear vector group F and K is a closed subgroup of H.

Because of 3.6, it is sufficient to show that F is a Schwartz group.

As explained above, for every absolutely convex closed neighborhood U of 0 in F there exists a absolutely convex closed neighborhood V such that $V \subseteq U$ and $\widetilde{\varphi_{VU}}$ is a nuclear operator. It is well known that every nuclear operator is precompact (see e.g. (3.1.5) in [24]). An easy argument shows that F is a locally quasi–convex Schwartz group. □

Example 4.4. *Let*

$$E = \{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_k := \sum_n |x_n|(n+1)^{-\frac{1}{k}} < \infty,\ \forall k \in \mathbb{N}\}.$$

Then:

(i) The vector space E with the topology τ determined by the sequence of norms $\|\cdot\|_k$, $k = 1, 2, \dots$ is a Fréchet space.

(ii) (E, τ) is a Schwartz space.

(iii) (E, τ) is not nuclear.

Proof. Proof. Let $\alpha_n := \ln(1+n)$, $n = 1, 2, \dots$ and $\alpha := (\alpha_n)_{n\in\mathbb{N}}$. Then with the notation in [18, p. 27 and p. 211] we have that $E = \Lambda_1(\alpha)$. Consequently, E is *a power series space of finite type.*

(i) follows from [18, p. 50 and p. 69 (Proposition 3.6.2)].

(ii) According to [18, p. 212], this follows from [18, p. 210 (Proposition 10.6.8)].

(iii) This follows from [18, p. 497 (Proposition 21.6.3)]. □

Another class of non nuclear Schwartz spaces will be presented in 5.7.

Theorem 4.5. *A Schwartz group is not necessarily locally quasi–convex. Even more: There exists a metrizable Schwartz group with trivial character group.*

Proof. Let F be a Fréchet Schwartz space which is not nuclear (4.4). According to a result of M. and W. Banaszczyk ([4]), there is a (discrete) closed subgroup H of F such that the quotient group $\mathrm{sp}(H)/H$, which is a Schwartz group by Proposition 3.6 (a.3), has no nontrivial continuous characters. This group is a non locally quasi–convex metrizable Schwartz group. □

5 Some classes of Schwartz groups

From now on, X will denote a completely regular Hausdorff space, and $C_{\mathrm{co}}(X)$ the space of real-valued continuous functions on X, endowed with the compact-open topology.

Recall that X is said to be *hemicompact* if it has a countable cobase of compact sets, i. e. there exists a sequence (K_n) of compact subsets of X

such that any compact subset of X is contained in one of them. On the other hand, X is said to be a *k-space* if a subset of X is open provided that its intersection with every compact subset is open with respect to the compact subset. In the literature hemicompact k-spaces are often called *k_ω-spaces* (see [11] for a survey of results concerning this class).

Definition 5.1. *X is said to* satisfy the Ascoli theorem *if every compact subset of $C_{\mathrm{co}}(X)$ is equicontinuous.*

The famous Arzela–Ascoli theorem states that k-spaces fulfill this property. However, in [22, p. 403] it is proved that arbitrary products of Čech complete spaces satisfy the Ascoli theorem as well. Consequently the class of spaces satisfying the Ascoli theorem is strictly wider than the class of k-spaces. (E.g. $\mathbb{Z}^I$ is not a k-space when I is an uncountable index set, see [19, Problem 7J(b)].)

In order to obtain new examples of Schwartz groups, we will use some results concerning free locally convex spaces and free topological Abelian groups in the sense of Markov ([20]). Recall that for a completely regular Hausdorff space X, the free Abelian topological group over X is the free Abelian group $A(X)$ endowed with the unique Hausdorff group topology for which the mapping $\eta : X \to A(X)$, which maps the topological space X onto a basis of $A(X)$, becomes a topological embedding and such that for every continuous mapping $f : X \to G$, where G is an Abelian Hausdorff group, the unique group homomorphism $\widetilde{f} : A(X) \to G$ which satisfies $f = \widetilde{f} \circ \eta$, is continuous.

Replacing topological Abelian groups by locally convex spaces and group homomorphisms by linear maps we obtain the definition of the free locally convex space $L(X)$ over X.

For an overview of this theory we refer the reader to the surveys [23], [28], and [9].

Our next theorem is based on the fact that the dual of a metrizable locally convex space endowed with the compact open topology is a locally convex Schwartz space (in [18, 16.4.2] it is proved that the dual of a metrizable space with the topology of precompact convergence is a Schwartz space and from [18, 9.4.2] it follows that on the dual of a metrizable space the topology of precompact convergence coincides with the compact-open topology).

Theorem 5.2. *If a hemicompact space X satisfies the Ascoli theorem then $L(X)$ is a Schwartz locally convex vector space.*

Proof. The mapping

$$I : L(X) \to C_{\mathrm{co}}(X)^*_{\mathrm{co}}, \ \ \sum \lambda_x x \mapsto [f \mapsto \sum \lambda_x f(x)]$$

is well defined, since $L(X)$ is algebraically free over X and the linear functionals $f \mapsto f(x)$ are continuous for the compact-open topology on $C_{\mathrm{co}}(X)$. The injectivity of I follows from the fact that X is completely regular.

According to [25], the topology on $L(X)$ is that of uniform convergence on all equicontinuous and pointwise bounded subsets of X. But, since X satisfies the Ascoli theorem, these are exactly the relatively compact subsets of $C_{\text{co}}(X)$. Hence I is an embedding.

Since X is hemicompact, $C_{\text{co}}(X)$ is a metrizable locally convex space. Hence $C_{\text{co}}(X)^*_{\text{co}}$ is a Schwartz space. Since $L(X)$ is a topological subspace of $C_{\text{co}}(X)^*_{\text{co}}$, it is a Schwartz space itself. □

Theorem 5.2 is a rather natural statement in locally convex space theory which does not seem to have been observed until now. Its analogue for groups is also true:

Corollary 5.3. *If a hemicompact space X satisfies the Ascoli theorem then $A(X)$ is a locally quasi-convex Schwartz group.*

Proof. For any Hausdorff completely regular space X, $A(X)$ is a topological subgroup of $L(X)$ ([28, Th. 3]). By Theorem 5.2, $L(X)$ is a Schwartz locally convex space, hence a Schwartz locally quasi-convex group, so is its subgroup $A(X)$. □

Theorem 5.4. *Every Hausdorff group G which is a hemicompact space and satisfies the Ascoli theorem, is a Schwartz group.*

Proof. According to Theorem 5.3, $A(G)$ is a Schwartz group. Moreover, the canonical mapping $A(G) \to G$ is a quotient mapping ([20] or (12.7) in [1]), thus G is a Schwartz group. □

Corollary 5.5. *Every Hausdorff group which is a hemicompact k-space is a Schwartz group.*

Corollary 5.6. *Let G be a metrizable group. Its character group $G^\wedge$ is a Schwartz group.*

Proof. It follows from 5.5 and the fact that the character group of a metrizable group is a hemicompact k-space (cf. (4.7) in [1] or Theorem 1 in [8]). □

Remark 5.7. *(a) In [3] it is shown that for a compact space X the free Abelian topological group $A(X)$ is a nuclear group if and only if X is finite. So for an infinite compact set X, the groups $A(X)$ and $L(X)$ are locally quasi–convex Schwartz groups but not nuclear.*

(b) Let X be a completely regular space. The group $C(X, \mathbb{T})$ of all continuous functions of X into $\mathbb{T}$, endowed with the compact–open topology, is a Schwartz group if and only if all compact subsets of X are finite. [Taking into account the permanence properties given in Section 3 above, we can replace “nuclear” by “Schwartz” in the statement and the proof of the analogous result involving nuclear groups ([1, 20.31]).]

The analogous result for the space of real valued functions $C(X, \mathbb{R})$ is contained in [18, Th. 10.8.1].

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