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Topologies on the direct sum of topological Abelian groups

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Abstract

We prove that the asterisk topologies on the direct sum of topological Abelian groups, used by Kaplan and Banaszczyk in duality theory, are different. However, in the category of locally quasi-convex groups they do not differ, and coincide with the coproduct topology.

MSC: 22A05

Keywords: Topological Abelian group; Locally quasi-convex group; Direct sum; Asterisk topology; Coproduct topology

1. Introduction

For a family, G_i, $i \in I$, in the category of topological Abelian groups, different topologies can be considered on their product $\prod_{i\in I} G_i$ or their direct sum $\bigoplus_{i\in I} G_i$, depending on the aim. The situation on the direct sum is intriguing, at least for uncountable families of groups.

Kaplan defined in [9] the *asterisk* topology $\mathcal{T}_{*_K}$, to obtain a duality principle between products and direct sums. He pointed out in his paper that, for countable direct sums, the asterisk topology coincide with the topology induced by the rectangular topology of the product. Since then, different authors (Noble [10], Venkataraman [14], Varopoulos [13],

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etc.) have used the asterisk topology on the direct sum of topological groups in the context of duality. Banaszczyk gave a new definition of the asterisk topology in [2], and proved a similar duality result but, as we see in the present paper, both definitions are not equivalent. They coincide in the category of locally quasi-convex groups, and the result of Kaplan was addressed for reflexive groups, which are within the mentioned category.

Another point of view is to consider the direct sum of a family of Abelian topological groups as the algebraic coproduct of the family. In this case the natural topology is named *coproduct* topology $\mathcal{T}_f$. It is the final group topology with respect to the family of canonical monomorphisms $\upsilon_j : G_j \to \bigoplus_{i\in I} G_i$. The topology $\mathcal{T}_f$ was considered by Higgins in a paper [7], where he describes some other topologies on the direct sum (including the asterisk topology of Kaplan). He proved in the same paper that the coproduct topology is not in general the direct limit of the topologies on the finite sums. Nickolas has taken recently [11] the work of Higgins and has found necessary and sufficient conditions for the coincidence of the considered topologies.

We show in this paper that in the category of locally quasi-convex groups the asterisk and coproduct topologies are closely connected: the first one is the locally quasi-convex topology associated to the second one.

2. Some inequalities in $\mathbb{T}$

We will use the following notations (the second one is due to Kaplan): For a subset V of an Abelian group G, we write

$$V_{(n)} = \{x \in G\colon\ x, 2x, \ldots, nx \in V\}, \quad n \in \mathbb{N},$$
$$\left(1/2^n\right)V = \left\{x \in V\colon\ 2^k x \in V\ \forall k \in \{0, 1, \ldots, n\}\right\}, \quad n \in \mathbb{N} \cup \{0\}.$$

$\mathbb{T}$ denotes the multiplicative group of complex numbers with modulus 1 endowed with its usual topology.

Given any $t \in \mathbb{T}$, $\theta(t)$ is the real number characterized by

$$\theta(t) \in]-1/2, 1/2], \quad \exp\bigl(2\pi i\theta(t)\bigr) = t.$$

We also denote

$$\mathbb{T}[\alpha, \beta] = \bigl\{\exp(2\pi i\lambda)\colon\ \lambda \in [\alpha, \beta]\bigr\}, \quad \alpha, \beta \in \mathbb{R},\ \alpha \leqslant \beta; \quad \mathbb{T}_+ = \mathbb{T}\left[-\frac{1}{4}, \frac{1}{4}\right].$$

Proposition 1. *Let* $\beta \in [0, 1/3[$. *Let* $t_1, \ldots, t_n \in \mathbb{T}[-\beta, \beta]$. *If*

$$\prod_{j=1}^{n} t_j^{\varepsilon_j} \in \mathbb{T}[-\beta, \beta] \quad \forall(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 0, 1\}^n,$$

then $|\theta(t_1)| + |\theta(t_2)| + \cdots + |\theta(t_n)| \leqslant \beta$.

Proof. We can suppose $\theta(t_j) \geqslant 0$ for every $j \in \{1, \ldots, n\}$. We will prove this result by induction. For $n = 1$ the assertion is trivial. Let us suppose that it is true for $n - 1$, and fix $t_1, \ldots, t_n$ with $\theta(t_j) \in [0, \beta]$ for every j in $\{1, \ldots, n\}$ and satisfying the above hypothesis.

Consider $t = t_1 t_2 \cdots t_{n-1}$. The family $t_1, t_2, \ldots, t_{n-1}$ also satisfy $\prod_{j=1}^{n-1} t_j^{\varepsilon_j} \in \mathbb{T}[-\beta, \beta]$ for every $(\varepsilon_1, \ldots, \varepsilon_{n-1})$ in $\{-1, 0, 1\}^{n-1}$, since we can take $\varepsilon_n = 1$ above; hence, applying the induction hypothesis we deduce $\theta(t_1) + \theta(t_2) + \cdots + \theta(t_{n-1}) \leqslant \beta$, and in particular $\theta(t) = \theta(t_1) + \theta(t_2) + \cdots + \theta(t_{n-1})$. We have

$$t t_n = \exp\bigl(2\pi i \bigl(\theta(t) + \theta(t_n)\bigr)\bigr) \in \mathbb{T}[-\beta, \beta]$$

and hence,

$$\begin{aligned}\theta(t_1) + \cdots + \theta(t_{n-1}) + \theta(t_n) &= \theta(t) + \theta(t_n)\\ &\in \bigl([-\beta, \beta] + \mathbb{Z}\bigr) \cap [0, 2\beta] = [0, \beta]. \qquad \square\end{aligned}$$

From Proposition 1 we obtain

Corollary 2.

(a) *For any* $n \in \mathbb{N}$ *and* $\beta \in [0, 1/3[$, $\mathbb{T}[-\beta, \beta]_{(n)} = \mathbb{T}[-\beta/n, \beta/n]$. *In particular* $(\mathbb{T}_+)_{(n)} = \mathbb{T}[-1/4n, 1/4n]$.
(b) *For any* $n \in \mathbb{N} \cup \{0\}$ *and* $\beta \in [0, 1/3[$, $(1/2^n)\mathbb{T}[-\beta, \beta] = \mathbb{T}[-\beta/2^n, \beta/2^n]$. *In particular* $(1/2^n)\mathbb{T}_+ = (\mathbb{T}_+)_{(2^n)}$.

Proof. (a) The inclusion $\mathbb{T}[-\beta/n, \beta/n] \subset \mathbb{T}[-\beta, \beta]_{(n)}$ is trivial. For the reverse inclusion, given $t \in \mathbb{T}[-\beta, \beta]_{(n)}$, we apply Proposition 1 with $t_j = t$ for every $j \in \{1, \ldots, n\}$; we deduce $n|\theta(t)| \leqslant \beta$ and thus $t \in \mathbb{T}[-\beta/n, \beta/n]$.

(b) Again, the inclusion $\mathbb{T}[-\beta/2^n, \beta/2^n] \subset (1/2^n)\mathbb{T}[-\beta, \beta]$ is trivial. The proof of the reverse inclusion can be made by induction. For $n = 0$ it is trivial. If we suppose that the assertion is true for $n - 1$, and fix $t \in (1/2^n)\mathbb{T}[-\beta, \beta]$, in particular we have

$$\begin{aligned}&t, t^2, t^{2^2}, \ldots, t^{2^{n-1}} \in \mathbb{T}[-\beta, \beta],\\ &t^2, \bigl(t^2\bigr)^2, \bigl(t^2\bigr)^{2^2}, \ldots, \bigl(t^2\bigr)^{2^{n-1}} \in \mathbb{T}[-\beta, \beta]\end{aligned}$$

and by hypothesis, t and t^2 both are in $\mathbb{T}[-\beta/2^{n-1}, \beta/2^{n-1}]$. From (a) we deduce $t \in \mathbb{T}[-\beta/2^n, \beta/2^n]$. $\square$

3. The functionals $(\cdot/U)$ and $(\cdot/U)_K$

We include the definitions of the two functionals $(\cdot/U)$ and $(\cdot/U)_K$, which are generalizations of the well-known Minkowski functional to groups. These two functionals take part in the definition of asterisk topologies on the direct sum.

Let G be an Abelian group and U a nonempty subset of G. The functional $(\cdot/U)_K$ was defined by Kaplan [9] in the following way:

$$(x/U)_K := \inf\left\{\frac{1}{2^n}\colon\ 2^k x \in U\ \forall k \in \{0, \ldots, n\}\right\}, \quad \forall x \in U. \tag{1}$$

Note that $(\cdot/U)_K$ takes its values in $\{0\} \cup \{1/2^n\colon\ n \in \mathbb{N} \cup \{0\}\}$, and that for every $n \in \mathbb{N} \cup \{0\}$, $(1/2^n)U = \{x \in U\colon\ (x/U)_K \leqslant 1/2^n\}$.

In [2, p. 8] Banaszczyk introduced the following variant:

$$(x/U) := \inf\left\{\frac{1}{n}\colon\ kx \in U\ \forall k \in \{1, \dots, n\}\right\}, \quad \forall x \in U. \tag{2}$$

The functional $(\cdot/U)$ takes its values in $\{0\} \cup \{1/n\colon\ n \in \mathbb{N}\}$. For any $n \in \mathbb{N}$, $U_{(n)} = \{x \in U\colon\ (x/U) \leqslant 1/n\}$.

Next we will prove a natural inequality involving $(\cdot/U)$ and $(\cdot/U)_K$:

Proposition 3. *For every Abelian group G, $U \subset G$ and $x \in U$*

$$(x/U)_K \leqslant 2(x/U).$$

Proof. Clearly this inequality is fulfilled in the case $(x/U) = 0$. Suppose now that $kx \in U$ for every $k \in \{1, \dots, n\}$ but $(n+1)x \notin U$. Since $2^{[\log_2 n]} \leqslant n$, we have

$$(x/U)_K \leqslant \frac{1}{2^{[\log_2 n]}} < \frac{2}{2^{\log_2 n}} = \frac{2}{n} = 2(x/U)$$

(here $[\alpha]$ stays for the greatest integer less or equal than the real number α). □

We cannot give a similar inequality (valid in general) in the opposite sense: if we put $G = \mathbb{R}$, $U = \{2^n\colon\ n \in \mathbb{N}\}$ and $x = 2$, it is immediate that $(x/U) = 1/2$ while $(x/U)_K = 0$.

The following proposition (cf. Lemma 1.14 in [2]) collects some other properties of $(\cdot/U)$; its proof is immediate.

Proposition 4. *Let G be an Abelian group and V, U nonempty subsets of G.*

(a) *If $V \subset U$, then $(x/U) \leqslant (x/V)$ for every $x \in V$.*
(b) *If $V + V \subset U$, then $(x/U) \leqslant \frac{1}{2}(x/V)$ for every $x \in V$.*
(c) *If $V + V \subset U$, then $(x + y/U) \leqslant \max\{(x/V), (y/V)\}$ for every $x, y \in V$.*

The same properties are true for Kaplan's functional $(\cdot/U)_K$.

4. Description of the final and box topologies

Let I be a nonempty set, G an Abelian group, G_i, $i \in I$ topological Abelian groups and $v_i : G_i \to G$, $i \in I$ group homomorphisms. In general the finest topology in G making all v_i continuous is not a group topology; examples of this situation can be found in [6, Exercises 2.10.21, 2.10.22], and [5, Example 2.9(a)]. However, it can be shown easily that the family of group topologies on G which make all the v_i continuous contains its supremum. The group topology obtained in this way is usually referred as the *final* topology on G corresponding to the family of homomorphisms $\{v_i\colon\ i \in I\}$.

We next give a neighborhood basis of zero for the final topology on G, following a construction which is known for the topological vector space case (see e.g., [8, 4.1]). In what follows, for a topological Abelian group G, $\mathcal{N}_0(G)$ will denote the family of all neighborhoods of zero in G.

Proposition 5. *Let I be a nonempty set, G an Abelian group, G_i, $i \in I$ topological Abelian groups and $v_i : G_i \to G$, $i \in I$ group homomorphisms. Let $\mathcal{T}_f$ denote the final topology on G corresponding to the family of homomorphisms $\{v_i\colon i \in I\}$. We define*

$$\mathsf{U}_f := \bigcup_{N\in\mathbb{N}} \bigcup_{(i_1,\dots,i_N)\in I^N} \sum_{n=1}^{N} v_{i_n}(U_{i_n,n}),$$

$$\mathsf{U} = (U_{i,n}) \in \prod_{i\in I} \mathcal{N}_0(G_i)^{\mathbb{N}}.$$

The family $\{\mathsf{U}_f\colon \mathsf{U} = (U_{i,n}) \in \prod_{i\in I} \mathcal{N}_0(G_i)^{\mathbb{N}}\}$ is a neighborhood basis of zero for the topology $\mathcal{T}_f$.

Proof. Let us first clarify the definition of the sets U_f: given a family of sequences of neighborhoods $(U_{i,n})$, $y \in G$ is in U_f if and only if there exist $N \in \mathbb{N}$ and $i_1,\dots,i_N \in I$ such that $y = v_{i_1}(x_1) + \cdots + v_{i_N}(x_N)$, being $x_n \in U_{i_n,n}$ for every $n \in \{1,\dots,N\}$.

The above family fulfills the axioms of neighborhood basis for a group topology; only some proof is needed to show that given $\mathsf{U} = (U_{i,n}) \in \prod_{i\in I} \mathcal{N}_0(G_i)^{\mathbb{N}}$ there exists $\mathsf{V} = (V_{i,n}) \in \prod_{i\in I} \mathcal{N}_0(G_i)^{\mathbb{N}}$ such that $\mathsf{V}_f + \mathsf{V}_f \subset \mathsf{U}_f$. Once fixed the neighborhoods $U_{i,n}$, we take for every $i \in I$ and $n \in \mathbb{N}$, some $V_{i,n} \in \mathcal{N}_0(G_i)$ satisfying $V_{i,n} \subset U_{i,2n-1} \cap U_{i,2n}$. Given y and y' in V_f, we have

$$\exists i_1,\dots,i_N \in I \text{ such that } y = v_{i_1}(x_1) + \cdots + v_{i_N}(x_N), \quad x_k \in V_{i_k,k},$$

$$\exists i'_1,\dots,i'_{N'} \in I \text{ such that } y' = v_{i'_1}(x'_1) + \cdots + v_{i'_{N'}}(x'_{N'}), \quad x'_l \in V_{i'_l,l}.$$

We can suppose $N = N'$ adding zeros if necessary. Hence

$$y + y' = v_{i_1}(x_1) + v_{i'_1}(x'_1) + \cdots + v_{i_N}(x_N) + v_{i'_{N'}}(x'_{N'})$$

being $x_k \in V_{i_k,k} \subset U_{i_k,2k-1}$, $x'_l \in V_{i'_l,l} \subset U_{i_l,2l}$. Renaming indices we obtain

$$y + y' = \sum_{n=1}^{2N} v_{j_n}(z_n), \quad z_n \in U_{j_n,n}$$

and hence, $y + y' \in \mathsf{U}_f$.

Now we prove that $\mathcal{T}_f$ is the finest group topology making all v_i continuous. Given $\mathsf{U} = (U_{i,n}) \in \prod_{i\in I} \mathcal{N}_0(G_i)^{\mathbb{N}}$ and $j \in I$, $v_j(U_{j,1}) \subset \mathsf{U}_f$ for every $j \in I$, so v_j is continuous. Let now τ be a group topology making all v_i continuous. Given a τ-neighborhood of zero W in G, we fix a sequence $(W_n)_{n\in\mathbb{N}}$ of τ-neighborhoods of zero in G such that $W_1 + \cdots + W_n \subset W$ for every $n \in \mathbb{N}$ (this can be done inductively in such a way that $W_1 + W_1 \subset W$, $W_2 + W_2 \subset W_1$, etc.). If we select $U_{i,n} \in \mathcal{N}_0(G_i)$ such that $v_i(U_{i,n}) \subset W_n$, for every $n \in \mathbb{N}$ and $i \in I$, we have $\mathsf{U}_f \subset W$. □

Remark 6. It can be proved without great effort that the above introduced basis of neighborhoods of zero for $\mathcal{T}_f$ can be modified in such a way that for every element in each neighborhood, the indices which take part in its decomposition are all different.

Next we will consider another important topology on the range G of a family of group homomorphisms $v_i : G_i \to G$.

In what follows, $\mathfrak{F}(I)$ will stay for the family of all finite subsets of a set I.

Proposition 7. *With the above notations, we define*

$$\boldsymbol{U}_b = \bigcup_{\Delta \in \mathfrak{F}(I)} \sum_{i \in \Delta} v_i(U_i),$$
$$\boldsymbol{U} = (U_i)_{i \in I} \in \prod_{i \in I} \mathcal{N}_0(G_i).$$

The family $\{\boldsymbol{U}_b\colon \boldsymbol{U} = (U_i)_{i\in I} \in \prod_{i\in I} \mathcal{N}_0(G_i)\}$ is a neighborhood basis of zero for a group topology $\mathcal{T}_b$ on G, which makes all v_i continuous.

If I is countable, $\mathcal{T}_b$ and $\mathcal{T}_f$ coincide on G.

Proof. It is easily checked that the given family is a basis of neighborhoods of zero for a group topology on G which makes all v_i continuous; in particular it is coarser than $\mathcal{T}_f$.

If I is countable, we can suppose $I \subset \mathbb{N}$. We will show that $\mathcal{T}_b$ is the finest group topology for which all v_i are continuous: Let $\mathcal{T}$ be a group topology on G making all v_i continuous. Given a $\mathcal{T}$-neighborhood of zero W in G, let (V_n) be a sequence in $\mathcal{N}_0(G)$ such that $V_1 + \cdots + V_n \subset W$ for every $n \in \mathbb{N}$. If $U_i \in \mathcal{N}_0(G_i)$ satisfies $v_i(U_i) \subset V_i$ for all $i \in I$, we have

$$\forall \Delta \in \mathfrak{F}(I) \quad \sum_{i\in\Delta} v_i(U_i) \subset \sum_{i\in\Delta} V_i \subset W \implies \boldsymbol{U}_b \subset W. \qquad \square$$

Remark 8. In [3] (Corollary to Proposition 1) it is proved that the topologies $\mathcal{T}_f$ and $\mathcal{T}_b$ coincide on any countable direct sum of locally compact groups.

We call $\mathcal{T}_b$ *box topology* (or *rectangular* topology) on G associated with the family of homomorphisms $\{v_i : G_i \to G\}$. This name is usually given to the topology $\mathcal{T}_b$ when the construction is carried out on the direct sum of a family of topological Abelian groups.

5. Topologies on the direct sum

Coproduct and box topologies on the direct sum. Let G_i be a topological Abelian group, for each $i \in I$. We may consider on the Abelian group $\bigoplus_{i\in I} G_i$ the final topology $\mathcal{T}_f$ (Proposition 5) or the box topology $\mathcal{T}_b$ (Proposition 7) with respect to the family of canonical monomorphisms $v_j : G_j \to \bigoplus_{i\in I} G_i$. We will use the denomination *coproduct topology* for the final topology $\mathcal{T}_f$ associated to this concrete system of groups and homomorphisms. We will use the notation $\bigoplus_{i\in I}^{(f)} G_i$ for the topological group $(\bigoplus_{i\in I} G_i, \mathcal{T}_f)$; and $\bigoplus_{i\in I}^{(b)} G_i$ for $(\bigoplus_{i\in I} G_i, \mathcal{T}_b)$. If U_i is a neighborhood of zero in G_i for each $i \in I$, we will write $\bigoplus_{i\in I} U_i$ for the neighborhood of zero $\mathbf{U}_b$ (Proposition 7).

Proposition 9. *Let $\mathcal{T}$ be a group topology on $\bigoplus_{i\in I} G_i$ such that $\mathcal{T}_b \subset \mathcal{T} \subset \mathcal{T}_f$. Then, for every $j \in I$, the natural homomorphism $\upsilon_j : G_j \to (\bigoplus_{i\in I} G_i, \mathcal{T})$ is a topological embedding.*

Proof. Fix $j \in I$. It is immediate that υ_j is one to one and (since $\mathcal{T} \subset \mathcal{T}_f$) continuous; let us show that it is relatively open. Let $U \in \mathcal{N}_0(G_j)$. If we take $(U_i)_{i\in I} \in \prod_{i\in I} \mathcal{N}_0(G_i)$ such that $U_j = U$, clearly $\upsilon_j(G_j) \cap \bigoplus_{i\in I} U_i \subset \upsilon_j(U)$. Since $\bigoplus_{i\in I} U_i$ is a $\mathcal{T}$-neighborhood of zero, the assertion follows. □

Corollary 10. *$\mathcal{T}_f$ is the finest group topology on $\bigoplus_{i\in I} G_i$ which induces the original topologies on all groups G_i.*

The asterisk topologies. Let G_i be an Abelian group and U_i a nonempty subset of G_i for each $i \in I$. We define

$$\bigoplus_{i\in I}^{(*_K)} U_i = \left\{ x \in \bigoplus_{i\in I} U_i \colon \sum_{i\in I} \big(x(i)/U_i\big)_K < 1 \right\},$$

$$\bigoplus_{i\in I}^{(*)} U_i = \left\{ x \in \bigoplus_{i\in I} U_i \colon \sum_{i\in I} \big(x(i)/U_i\big) < 1 \right\}.$$

Using basic properties of the functional $(\cdot/U)$ (Proposition 4), and their analogues for $(\cdot/U)_K$, it is easy to check that both the families formed by the sets $\bigoplus_{i\in I}^{(*_K)} U_i$ and $\bigoplus_{i\in I}^{(*)} U_i$, as $(U_i)_{i\in I}$ runs over $\prod_{i\in I} \mathcal{N}_0(G_i)$, satisfy the axioms corresponding to a neighborhood basis of zero for a group topology. We will denote these topologies respectively by $\mathcal{T}_{*_K}$ and $\mathcal{T}_*$. The first one was introduced by Kaplan in [9], and the second one was defined by Banaszczyk in [2]. Both authors used the term *asterisk topology* for their definitions. We shall keep this denomination only for $\mathcal{T}_*$.

We will denote the topological groups $(\bigoplus_{i\in I} G_i, \mathcal{T}_{*_K})$ and $(\bigoplus_{i\in I} G_i, \mathcal{T}_*)$ respectively by $\bigoplus_{i\in I}^{(*_K)} G_i$ and $\bigoplus_{i\in I}^{(*)} G_i$.

Comparison of topologies. The following results collect some facts about the relations between the different topologies above defined on the direct sum. We will use the preceding notations, and also denote by $\mathcal{T}_\pi$ the topology on the direct sum induced by the Tychonov topology.

Proposition 11. *Let $(G_i)_{i\in I}$ be a family of topological Abelian groups. On the direct sum $\bigoplus_{i\in I} G_i$,*

(a) $\mathcal{T}_\pi \subset \mathcal{T}_b \subset \mathcal{T}_{*_K} \subset \mathcal{T}_* \subset \mathcal{T}_f$.
(b) *If I is finite, then $\mathcal{T}_\pi = \mathcal{T}_b = \mathcal{T}_{*_K} = \mathcal{T}_* = \mathcal{T}_f$.*
(c) *If I is countable, then $\mathcal{T}_b = \mathcal{T}_{*_K} = \mathcal{T}_* = \mathcal{T}_f$.*

Proof. (a) Only the inclusion $\mathcal{T}_{*_K} \subset \mathcal{T}_*$ needs proof: Let U_i be a neighborhood of zero in G_i, for every $i \in I$. We will prove that $\bigoplus_{i\in I}^{(*)} V_i \subset \bigoplus_{i\in I}^{(*_K)} U_i$ for any $(V_i)_{i\in I} \in$

$\prod_{i\in I}\mathcal{N}_0(G_i)$ with $V_i+V_i\subset U_i,\ i\in I$. Indeed, if $x\in\bigoplus_{i\in I}^{(*)}V_i$, then $x(i)\in V_i\subset U_i$ for every $i\in I$ and we have

$$\sum_{i\in I}\bigl(x(i)/U_i\bigr)_K\overset{\text{Prop. 3}}{\leqslant}2\sum_{i\in I}\bigl(x(i)/U_i\bigr)\overset{\text{Prop. 4(b)}}{\leqslant}\sum_{i\in I}\bigl(x(i)/V_i\bigr)<1.$$

(b) It is immediate that for I finite, any group topology on $\bigoplus_{i\in I}G_i$ which induces original topologies on all G_i is coarser than the product topology. In particular $\mathcal{T}_f\subset\mathcal{T}_\pi$ and we deduce that all these topologies coincide.

(c) This follows from the fact that in general $\mathcal{T}_b$ and $\mathcal{T}_f$ coincide for countable I (Proposition 7). □

Remark 12. Recently [11] Nickolas has found necessary and sufficient conditions for the coincidence between any two of the topologies $\mathcal{T}_b$, $\mathcal{T}_{*_K}$ and $\mathcal{T}_f$.

We devote the remaining of this section to investigate the relation between the topologies $\mathcal{T}_{*_K}$ and $\mathcal{T}_*$.

Proposition 13. *Let $\{G_i\colon i\in I\}$ be a family of topological Abelian groups. The following are equivalent*:

(a) $\mathcal{T}_{*_K}=\mathcal{T}_*$ *on* $\bigoplus_{i\in I}G_i$.
(b) *For all but countably many $i\in I$, given $U\in\mathcal{N}_0(G_i)$ there exists $V\in\mathcal{N}_0(G_i)$ such that $(1/2^n)V\subset U_{(2^n)}$ for every $n\in\mathbb{N}$.*

Proof. Suppose that (b) is not satisfied. Then clearly we can find an uncountable $I_1\subset I$ and a family of sets $(V_i)_{i\in I_1}$ such that for every $i\in I_1$, V_i is a neighborhood of zero in G_i and for each $W_i\in\mathcal{N}_0(G_i)$ there exists $N_i\in\mathbb{N}$ with $(1/2^{N_i})W_i\not\subset(V_i)_{(2^{N_i})}$. Fix $V_i\in\mathcal{N}_0(G_i)$ arbitrary for $i\in I\setminus I_1$ and let us show that $\bigoplus_{i\in I}^{(*)}V_i$ is not a $\mathcal{T}_{*_K}$-neighborhood of zero in $\bigoplus_{i\in I}G_i$. Clearly it suffices to prove that for any family $(W_i)_{i\in I_1}$ with $W_i\in\mathcal{N}_0(G_i)$ for every $i\in I_1$, $\bigoplus_{i\in I_1}^{(*_K)}W_i\not\subset\bigoplus_{i\in I_1}^{(*)}V_i$. Let N_i ($i\in I_1$) be the natural number associated with each W_i as above. Since I_1 is not countable, there exists $N\in\mathbb{N}$ such that $\{i\in I_1\colon N_i=N\}$ is infinite. Let Δ be a subset of this set with cardinal 2^N-1. We define $x\in\bigoplus_{i\in I_1}G_i$ in the following way: for every $i\in\Delta$, we take $x(i)$ in $(1/2^N)W_i$ but not in $(V_i)_{(2^N)}$; for every $i\notin\Delta$, we put $x(i)=0$. Then

$$\sum_{i\in I_1}\bigl(x(i)/V_i\bigr)=\sum_{i\in\Delta}\bigl(x(i)/V_i\bigr)\geqslant\bigl(2^N-1\bigr)\frac{1}{2^N-1}=1\implies x\notin\bigoplus_{i\in I_1}^{(*)}V_i,$$

$$\sum_{i\in I_1}\bigl(x(i)/W_i\bigr)_K=\sum_{i\in\Delta}\bigl(x(i)/W_i\bigr)_K\leqslant\bigl(2^N-1\bigr)\frac{1}{2^N}<1\implies x\in\bigoplus_{i\in I_1}^{(*_K)}W_i.$$

Suppose that (b) is satisfied, and let us prove (a). Let I_0 be a countable subset of I such that, for every $i\in I\setminus I_0$, the neighborhoods of zero in G_i satisfy the condition in (b).

Fix a basic $\mathcal{T}_*$-neighborhood $\bigoplus_{i\in I}^{(*)} U_i$ in $\bigoplus_{i\in I} G_i$. We will find a family $(W_i)_{i\in I}$ with $W_i \in \mathcal{N}_0(G_i)$ for every $i \in I$ and

$$\bigoplus_{i\in I}^{(*_K)} W_i \subset \bigoplus_{i\in I}^{(*)} U_i.$$

For every $i \in I$, let V_i be a neighborhood of zero in G_i such that $V_i + V_i \subset U_i$. Fix now $W_i \in \mathcal{N}_0(G_i)$ $(i \in I_0)$ such that $\bigoplus_{i\in I_0} W_i \subset \bigoplus_{i\in I_0}^{(*)} V_i$; this can be done since for countable index sets the topologies $\mathcal{T}_b$ and $\mathcal{T}_*$ coincide (Proposition 11(c)). For $i \in I \setminus I_0$, fix $W_i \in \mathcal{N}_0(G_i)$ such that $(1/2^n)W_i \subset (V_i)_{(2^n)}$ for every $n \in \mathbb{N}$.

Take $x \in \bigoplus_{i\in I}^{(*_K)} W_i$.

For every $i \in I \setminus I_0$, $(x(i)/V_i) \leqslant (x(i)/W_i)_K$. Indeed, if $(x(i)/W_i)_K = 0$, we have $x(i) \in (1/2^n)W_i \subset (V_i)_{(2^n)}$ for every $n \in \mathbb{N}$, and hence $(x(i)/V_i) = 0$. If $(x(i)/W_i)_K = 1/2^n$, then we have $x(i) \in (1/2^n)W_i \subset (V_i)_{(2^n)}$ and hence $(x(i)/V_i) \leqslant 1/2^n = (x(i)/W_i)_K$. We deduce

$$\sum_{i\in I}\bigl(x(i)/U_i\bigr) \overset{\text{Prop. 4(c)}}{\leqslant} \frac{1}{2}\sum_{i\in I}\bigl(x(i)/V_i\bigr) \leqslant \frac{1}{2}\biggl(\sum_{i\in I_0}\bigl(x(i)/V_i\bigr) + \sum_{i\in I\setminus I_0}\bigl(x(i)/V_i\bigr)\biggr).$$

Since the family $(x(i))_{i\in I_0}$ is in $\bigoplus_{i\in I_0} W_i$ and hence, in $\bigoplus_{i\in I_0}^{(*)} V_i$, we have

$$\sum_{i\in I}\bigl(x(i)/U_i\bigr) < \frac{1}{2}\biggl(1 + \sum_{i\in I\setminus I_0}\bigl(x(i)/W_i\bigr)_K\biggr) < 1;$$

thus we have proved that $x = (x(i))_{i\in I}$ is in the $\mathcal{T}_*$-neighborhood of zero $\bigoplus_{i\in I}^{(*)} U_i$. □

Next we show that the topologies $\mathcal{T}_*$ and $\mathcal{T}_{*_K}$ are in general different.

Consider the Abelian group $\mathbb{Z}$. In [12] the following fact is proved: For any $x \in \mathbb{Z}$ there exists a unique decomposition of x as $x = \sum a_n 2^n$, $a_n \in \{-1, 0, 1\}$ (the sum ranging over $\mathbb{N} \cup \{0\}$ and including only finitely many nonzero a_n) satisfying the condition

$$[n \geqslant 1,\ a_n \neq 0 \implies a_{n-1} = a_{n+1} = 0]. \tag{3}$$

For a decreasing and convergent to zero sequence of positive real numbers $\delta = (\delta_n)_{n\geqslant 0}$, we define the map

$$q_\delta : \mathbb{Z} \to [0, \infty[\,, \quad q_\delta\Bigl(\sum a_n 2^n\Bigr) = \sum |a_n|\delta_n.$$

In [12] it is proved that in fact, for every $x \in \mathbb{Z}$, $q_\delta(x)$ is the infimum of all sums $\sum |a_n|\delta_n$ where (a_n) is any eventually zero sequence in $\mathbb{Z}$ such that $x = \sum a_n 2^n$. Using this fact, it is easy to show that q_δ is a separated quasinorm, i.e., that for every $x, y \in \mathbb{Z}$

$$q_\delta(x) = 0 \iff x = 0; \qquad q_\delta(-x) = q_\delta(x); \qquad q_\delta(x+y) \leqslant q_\delta(x) + q_\delta(y).$$

Let τ_δ be the metrizable topology on $\mathbb{Z}$ associated with the pseudonorm q_δ, and $\mathbb{Z}_\delta$ the topological Abelian group $(\mathbb{Z}, \tau_\delta)$. For each $\varepsilon > 0$, B_ε denotes the set $q_\delta^{-1}[0, \varepsilon]$.

Lemma 14. *If we define, for every $n \in \mathbb{N}$,*

$$l_n = 2^0 + 2^2 + \cdots + 2^{2n} = \frac{4^{n+1} - 1}{3},$$

$$m_n = \sum_{k=1}^{n} l_k \quad (m_0 = 0),$$

$$\delta_j = 1/n \quad \text{for } m_{n-1} \leqslant j < m_n,$$

$$k_j = l_n \quad \text{for } m_{n-1} \leqslant j < m_n,$$

then $q_\delta(k_j 2^j) > 1\ \forall j \in \mathbb{N}$.

Proof. Fix $j \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $j \in \{m_{n-1}, \ldots, m_n - 1\}$. We have

$$q_\delta\bigl(k_j 2^j\bigr) = q_\delta\bigl(\bigl(2^0 + 2^2 + \cdots + 2^{2n}\bigr)2^j\bigr) = q_\delta\bigl(2^j + 2^{j+2} + \cdots + 2^{j+2n}\bigr),$$

which by definition of q_δ is exactly $\delta_j + \delta_{j+2} + \delta_{j+4} + \cdots + \delta_{j+2n} > (n+1)\frac{1}{n+1} = 1$. □

Proposition 15. *Let I be an uncountable set and for each $i \in I$, let G_i be the topological Abelian group $\mathbb{Z}_\delta$, with $\delta = (\delta_n)_{n\in\mathbb{N}}$ as in Lemma* 14. *Then the topology $\mathcal{T}_*$ on $\bigoplus_{i\in I} G_i$ is strictly finer than $\mathcal{T}_{*_K}$.*

Proof. By Proposition 13(b), it is sufficient to show that in the metric group $(\mathbb{Z}, q_\delta)$

$$\forall \varepsilon > 0\ \exists n \in \mathbb{N}, \qquad \bigl(1/2^n\bigr)B_\varepsilon \not\subset (B_1)_{(2^n)}.$$

It is immediate that for every $x \in \mathbb{Z}$ and $n \geqslant 0$, $q_\delta(2^n x) \leqslant q_\delta(x)$, and in particular $(1/2^n)B_\varepsilon = B_\varepsilon$ for every $n \in \mathbb{N}$ and $\varepsilon > 0$. Hence it suffices to prove that

$$\forall \varepsilon > 0\ \exists k \in \mathbb{N}\ \exists x \in \mathbb{Z}, \quad q_\delta(x) \leqslant \varepsilon, \quad q_\delta(kx) > 1.$$

Given $\varepsilon > 0$, take $j \in \mathbb{N}$ such that $\delta_j \leqslant \varepsilon$. Then $q_\delta(2^j) = \delta_j \leqslant \varepsilon$, but by Lemma 14, $q_\delta(k_j 2^j) > 1$. □

6. Duality and quasi-convexity of direct sums

Let G be a topological Abelian group. The name *character* of G is used for a group homomorphism $\chi : G \to \mathbb{T}$. We denote by $\mathrm{Hom}(G, \mathbb{T})$ the group of characters of G and by $G^\wedge$ the group of continuous characters of G; $G^\wedge$ is called the *dual group* of G. The group $G^\wedge$ is usually endowed with the compact open topology and the *bidual group* $G^{\wedge\wedge}$ is the dual group of $G^\wedge$. The group G is said to be *reflexive* when the canonical evaluation $\alpha_G : G \to G^{\wedge\wedge}$ given by $\alpha_G(x)(\kappa) = \kappa(x)$ is a topological isomorphism.

For a subset A of G, the *polar* of A is

$$A^\triangleright = \bigl\{\chi \in G^\wedge\colon\ \chi(A) \subset \mathbb{T}_+\bigr\}.$$

A character χ of the topological Abelian group G is continuous if and only if $\chi \in U^\triangleright$ for some neighborhood of zero U in G.

In the same way, for a subset B of $G^\wedge$ the *inverse polar* of B is

$$B^{\triangleleft} = \{x \in G\colon\ \chi(x) \in \mathbb{T}_+\ \forall \chi \in B\}.$$

The subset A of G is *quasi-convex* if $A^{\triangleright\triangleleft} = A$, that is, if it satisfies the following condition:

$$\forall x \in G \setminus A\ \exists \chi \in G^\wedge, \quad \chi(A) \subset \mathbb{T}_+, \quad \chi(x) \notin \mathbb{T}_+.$$

Polars and inverse polars are quasi-convex sets.

We say that a topological Abelian group is *locally quasi-convex* if it admits a neighborhood basis of zero formed by quasi-convex sets. Locally quasi-convex groups are a natural generalization of locally convex spaces, since by Hahn-Banach theorem, in any locally convex space the bipolars of all neighborhoods of zero form themselves a neighborhood basis. In fact the underlying group of any locally convex space is locally quasi-convex [2, Proposition 2.4]. Since polars are quasi-convex sets, for any topological Abelian group G its dual $G^\wedge$ is locally quasi-convex; in particular, any reflexive topological Abelian group is locally quasi-convex.

Proposition 16. *Let (G, τ) be a topological Abelian group. The family of sets $\{U^{\triangleright\triangleleft}\colon U \in \mathcal{N}_0(G)\}$ is a neighborhood basis of zero for a group topology τ_{qc} on G; τ_{qc} is the finest among those locally quasi-convex group topologies on G which are coarser than τ. The topological groups (G, τ) and (G, τ_{qc}) have the same continuous characters.*

Proof. This result is presented and proved in both the references [1, Proposition 6.18] and [4, 4.6]. □

Proposition 17. *Let $\{G_i\colon i \in I\}$ be a family of topological Abelian groups. If all but countably many groups in the family are locally quasi-convex, then the topologies $\mathcal{T}_{*K}$ and $\mathcal{T}_*$ coincide on $\bigoplus_{i\in I} G_i$.*

Proof. It suffices to prove (Proposition 13) that given a locally quasi-convex G and $U \in \mathcal{N}_0(G)$, there exists $V \in \mathcal{N}_0(G)$ such that $(1/2^n)V \subset U_{(2^n)}$ for every $n \in \mathbb{N}$. For that, take as V any quasi-convex neighborhood of zero contained in U. Let x be an element of G and n a natural number such that the finite sequence $x, 2x, 2^2x, \ldots, 2^n x$ is contained in V. We need to see that for all $k \in \{1, \ldots, 2^n\}$, $kx \in U$. It is enough to prove that $kx \in V^{\triangleright\triangleleft} = V \subset U$ for any such k. Fix any $\chi \in V^{\triangleright}$. The finite sequence $\chi(x), \chi(x)^2, \chi(x)^{2^2}, \ldots, \chi(x)^{2^n}$ is contained in $\mathbb{T}_+$, so (Corollary 2(b)) $\chi(x) \in (\mathbb{T}_+)_{(2^n)}$. Then $\chi(kx) = \chi(x)^k \in \mathbb{T}_+$. □

Remark 18. It is a well-known fact that for a family of topological Abelian groups $(G_i)_{i\in I}$, the canonical map between $(\prod_{i\in I} G_i, \mathcal{T}_\pi)^\wedge$ and $\bigoplus_{i\in I} G_i^\wedge$ becomes a topological isomorphism if we consider the compact-open topology on the domain and the topology $\mathcal{T}_{*K}$ on the range. This was firstly stated for reflexive G_i by Kaplan [9]; Banaszczyk [2] proved the same result for arbitrary G_i and considering the topology $\mathcal{T}_*$ on the sum of the dual groups. Note that the fact that both asterisk topologies coincide on this sum is an immediate consequence of Proposition 17, since dual groups are locally quasi-convex.

Lemma 19. *Let G be a topological Abelian group and U a subset of G.*

(a) *For every $x \in U$, $(x/U) \geqslant 4 \sup_{\chi \in U^{\triangleright}} |\theta(\chi(x))|$.*
(b) *If U is quasi-convex, then for every $x \in U$ $(x/U) \leqslant 8 \sup_{\chi \in U^{\triangleright}} |\theta(\chi(x))|$.*

Proof. (a) If $(x/U) = 0$, then $nx \in U$ and in particular $\chi(nx) = \chi(x)^n \in \mathbb{T}_+$ for every $n \in \mathbb{N}$ and $\chi \in U^{\triangleright}$. From Corollary 2(a) we deduce $\chi(x) = 1$ for every $\chi \in U^{\triangleright}$ and thus, $\sup_{\chi \in U^{\triangleright}} |\theta(\chi(x))| = 0$. Otherwise, $(x/U) = 1/N$ for some $N \in \mathbb{N}$. In particular $x, 2x, \ldots, Nx \in U$ and hence, $\chi(x), \chi(x)^2, \ldots, \chi(x)^N \in \mathbb{T}_+$ for any $\chi \in U^{\triangleright}$. Again by Corollary 2(a),

$$\theta\bigl(\chi(x)\bigr) \leqslant \frac{1}{4N} = \frac{1}{4}(x/U)$$

for any such χ, and the inequality follows.

(b) If $(x/U) = 0$ the inequality is trivial. Otherwise, $(x/U) = 1/N$ for some $N \in \mathbb{N}$; in particular $(N+1)x \notin U$. Since U is a quasi-convex set, we can find $\chi \in U^{\triangleright}$ with $\chi((N+1)x) = \chi(x)^{N+1} \notin \mathbb{T}_+$. Hence

$$\bigl|\theta\bigl(\chi(x)\bigr)\bigr| > \frac{1}{4(N+1)} \geqslant \frac{1}{8}(x/U)$$

and the inequality follows. □

Let $\{G_i\colon i \in I\}$ be a family of Abelian groups and $\bigoplus_{i \in I} G_i$ their algebraic direct sum. Clearly, for any character $\chi : \bigoplus_{i \in I} G_i \to \mathbb{T}$ there exist uniquely defined characters $\chi_i : G_i \to \mathbb{T}$, $i \in I$ such that $\chi(x) = \prod_{i \in I} \chi_i(x(i))$ for every $x \in \bigoplus_{i \in I} G_i$. Inversely, for any family $(\chi_i)_{i \in I}$ in $\prod_{i \in I} \mathrm{Hom}(G_i, \mathbb{T})$, the expression $x \mapsto \prod_{i \in I} \chi_i(x(i))$ defines a character of the Abelian group $\bigoplus_{i \in I} G_i$. According with this, in the following lemma we identify the Abelian group $\prod_{i \in I} \mathrm{Hom}(G_i, \mathbb{T})$ with the algebraic dual of $\bigoplus_{i \in I} G_i$; in other words, we consider $\bigoplus_{i \in I} G_i$ equipped with the discrete topology.

Lemma 20. *Let G_i, $i \in I$ be topological Abelian groups, and for every $i \in I$, let V_i, U_i be nonempty subsets of G_i.*

(a) $\prod_{i \in I} U_i^{\triangleright} \subset (\bigoplus_{i \in I}^{(*)} U_i)^{\triangleright}$.
(b) *If $V_i + V_i + V_i \subset U_i$ for every $i \in I$, then $(\prod_{i \in I} V_i^{\triangleright})^{\triangleleft} \subset \bigoplus_{i \in I}^{(*)} U_i^{\triangleright\triangleleft}$.*

Proof. (a) Fix $x \in \bigoplus_{i \in I}^{(*)} U_i$ and $(\chi_i)_{i \in I} \in \prod_{i \in I} U_i^{\triangleright}$. We denote by Δ the finite set formed by those $i \in I$ such that $x(i) \neq 0$. Let n_i $(i \in \Delta)$ be natural numbers such that $x(i) \in (U_i)_{(n_i)}$ for every $i \in \Delta$, and $\sum_{i \in \Delta} \frac{1}{n_i} < 1$. By Corollary 2(a), for every $i \in \Delta$ we have $\chi_i(x(i)) \in (\mathbb{T}_+)_{(n_i)} = \mathbb{T}[-1/4n_i, 1/4n_i]$, and hence $\prod_{i \in I} \chi_i(x(i)) = \prod_{i \in \Delta} \chi_i(x(i)) \in \mathbb{T}_+$.

(b) Let $x \in (\prod_{i\in I} V_i^{\triangleright})^{\triangleleft}$. Let us denote again by Δ the finite set formed by those $i \in I$ such that $x(i) \neq 0$. It is clear that $x(i) \in U_i^{\triangleright\triangleleft}$ for every $i \in I$; we must prove that $\sum_{i\in\Delta}(x(i)/U_i^{\triangleright\triangleleft}) < 1$. By Lemma 19(b)

$$\left(x(i)/U_i^{\triangleright\triangleleft}\right) \leqslant 8 \sup_{\chi\in U_i^{\triangleright\triangleleft\triangleright}} \left|\theta\left(\chi\left(x(i)\right)\right)\right| = 8 \sup_{\chi\in U_i^{\triangleright}} \left|\theta\left(\chi\left(x(i)\right)\right)\right| \quad \forall i \in \Delta.$$

Let now $\chi_i \in U_i^{\triangleright}$ for every $i \in I$. Fix $(\varepsilon_i)_{i\in I} \in \{-1, 0, 1\}^I$. Since $V_i + V_i + V_i \subset U_i$ for every $i \in I$, the families $(\chi_i^{k\varepsilon_i})_{i\in I}$, for $k \in \{1, 2, 3\}$, are in $\prod_{i\in I} V_i^{\triangleright}$ and since $x \in (\prod_{i\in I} V_i^{\triangleright})^{\triangleleft}$, we have

$$\left(\prod_{i\in\Delta} \chi_i^{\varepsilon_i}\left(x(i)\right)\right)^k \in \mathbb{T}_+, \quad k \in \{1, 2, 3\}.$$

By Corollary 2(a), $\prod_{i\in\Delta} \chi_i^{\varepsilon_i}(x(i)) \in (\mathbb{T}_+)_{(3)} = \mathbb{T}[-1/12, 1/12]$ and by Proposition 1,

$$\sum_{i\in\Delta}\left|\theta\left(\chi_i\left(x(i)\right)\right)\right| \leqslant \frac{1}{12}.$$

Since the characters χ_i were arbitrary, we deduce

$$\sum_{i\in\Delta}\left(x(i)/U_i^{\triangleright\triangleleft}\right) \leqslant 8\sum_{i\in\Delta}\sup_{\chi\in U_i^{\triangleright}}\left|\theta\left(\chi\left(x(i)\right)\right)\right| \leqslant 8\cdot\frac{1}{12} < 1. \qquad \square$$

Let now $(G_i)_{i\in I}$ be a nonempty family of *topological* Abelian groups, and let $\mathcal{T}$ be a group topology defined on the direct sum $\bigoplus_{i\in I} G_i$, *for which all maps* $v_i : G_i \to \bigoplus_{i\in I} G_i$ *are continuous* (i.e., $\mathcal{T} \subset \mathcal{T}_f$). Then we can define the canonical monomorphism

$$\Phi : \chi \in \left(\bigoplus_{i\in I} G_i, \mathcal{T}\right)^{\wedge} \mapsto (\chi \circ v_i)_{i\in I} \in \prod_{i\in I} G_i^{\wedge}.$$

Kaplan [9] proved that if the groups G_i are reflexive and $\mathcal{T} = \mathcal{T}_{*K}$ then Φ is a topological isomorphism, considering compact-open topology on duals and product topology on the product. Banaszczyk [2] proved the same for Hausdorff G_i and $\mathcal{T} = \mathcal{T}_*$. An easy adaptation of their proofs leads to the same result where $\mathcal{T}$ is any group topology on the direct sum which contains $\mathcal{T}_{*K}$; in particular, for $\mathcal{T} = \mathcal{T}_f$.

Theorem 21. *Let $(G_i)_{i\in I}$ be a family of locally quasi-convex topological Abelian groups. The asterisk topology $\mathcal{T}_*$ is the finest locally quasi-convex topology on $\bigoplus_{i\in I} G_i$ which makes the canonical injections continuous.*

Proof. $\mathcal{T}_*$ is locally quasi-convex: Let U_i be a quasi-convex neighborhood of zero in G_i for every $i \in I$. Let V_i be neighborhoods of zero in G_i such that $V_i + V_i + V_i \subset U_i$ for each $i \in I$. By Lemma 20(a), $\prod_{i\in I} V_i^{\triangleright} \subset (\bigoplus_{i\in I}^{(*)} V_i)^{\triangleright}$, hence $\bigoplus_{i\in I}^{(*)} V_i \subset (\bigoplus_{i\in I}^{(*)} V_i)^{\triangleright\triangleleft} \subset (\prod_{i\in I} V_i^{\triangleright})^{\triangleleft}$. Using now Lemma 20(b), we conclude that

$$\bigoplus_{i\in I}^{(*)} V_i \subset \left(\prod_{i\in I} V_i^{\triangleright}\right)^{\triangleleft} \subset \bigoplus_{i\in I}^{(*)} U_i.$$

$(\prod_{i\in I} V_i^{\triangleright})^{\triangleleft}$ is a quasi-convex subset of the discrete group $\bigoplus_{i\in I} G_i$ (recall that inverse polars are quasi-convex sets); moreover, it is a $\mathcal{T}_*$-neighborhood of zero, since it contains $\bigoplus_{i\in I}^{(*)} V_i$. Hence it is a quasi-convex neighborhood of zero for the asterisk topology.

$\mathcal{T}_*$ makes the υ_i continuous: this is an immediate consequence of Propositions 9 and 11.

$\mathcal{T}_*$ is the finest one under these conditions: Let $\mathcal{T}$ be a locally quasi-convex topology on $\bigoplus_{i\in I} G_i$ for which the monomorphisms υ_i are all continuous. Let U be a quasi-convex neighborhood of zero in $(\bigoplus_{i\in I} G_i, \mathcal{T})$. For every $i \in I$ there exists $U_i \in \mathcal{N}_0(G_i)$ such that $\upsilon_i(U_i) \subset U$. Let $U^{\triangleright}$ be the polar of U for the topology $\mathcal{T}$. Since $\mathcal{T}$ makes all υ_i continuous, any $\mathcal{T}$-continuous character can be identified with a family $(\chi_i)_{i\in I}$ in $\prod_{i\in I} G_i^{\wedge}$; since for every $i \in I$ we have $\upsilon_i(U_i) \subset U$, it is immediate that in fact $U^{\triangleright} \subset \prod_{i\in I} U_i^{\triangleright}$. We deduce that $(\prod_{i\in I} U_i^{\triangleright})^{\triangleleft} \subset U^{\triangleright\triangleleft} = U$, but $(\prod_{i\in I} U_i^{\triangleright})^{\triangleleft}$ is a $\mathcal{T}_*$-neighborhood of zero; indeed, by Lemma 20(a) it contains $\bigoplus_{i\in I}^{(*)} U_i$, as above. □

The fact that, for locally quasi-convex G_i, the asterisk topology on the direct sum of the family is locally quasi-convex, was mentioned in [2] (Proposition 1.16), and proved independently in [1] (Proposition 6.8 (iii)).

Corollary 22. *Let $(G_i)_{i\in I}$ be a family of locally quasi-convex topological Abelian groups. The asterisk topology on $\bigoplus_{i\in I} G_i$ is the locally quasi-convex topology associated with $\mathcal{T}_f$* (*in the sense of Proposition* 16).

Proof. We must show that under the above conditions, $\mathcal{T}_*$ is the finest locally quasi-convex group topology among those coarser than $\mathcal{T}_f$. We know that $\mathcal{T}_* \subset \mathcal{T}_f$; and that $\mathcal{T}_*$ is locally quasi-convex (Theorem 21). Let $\mathcal{T}$ be a locally quasi-convex group topology on $\bigoplus_{i\in I} G_i$ coarser than $\mathcal{T}_f$. Since $\mathcal{T} \subset \mathcal{T}_f$, $\mathcal{T}$ makes the canonical injections continuous. By Theorem 21, $\mathcal{T} \subset \mathcal{T}_*$. □

Corollary 23. *Let $(G_i)_{i\in I}$ be a family of locally quasi-convex topological Abelian groups. The topology $\mathcal{T}_f$ on $\bigoplus_{i\in I} G_i$ is locally quasi-convex if and only if it coincides with $\mathcal{T}_*$.*

Proof. This is an immediate consequence of Corollary 22. □

Nickolas [11] pointed out that on any direct sum of topological Abelian groups which topologically are P-spaces, the topologies $\mathcal{T}_*$ and $\mathcal{T}_f$ coincide. Such groups are locally quasi-convex, since they have a neighborhood basis of zero formed by open subgroups. This provides a nontrivial example of the situation characterized in Corollary 23.

Remark 24. Theorem 21 may be reformulated saying that the coproduct, in the category of locally quasi-convex groups, is the direct sum endowed with the asterisk topology. Hence in this category, the coproduct of reflexive groups is reflexive.

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