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# Completeness properties of locally quasi-convex groups

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**Abstract**

It is natural to extend the Grothendieck theorem on completeness, valid for locally convex topological vector spaces, to Abelian topological groups. The adequate framework to do it seems to be the class of locally quasi-convex groups. However, in this paper we present examples of metrizable locally quasi-convex groups for which the analogue to the Grothendieck theorem does not hold. By means of the continuous convergence structure on the dual of a topological group, we also state some weaker forms of the Grothendieck theorem valid for the class of locally quasi-convex groups. Finally, we prove that for the smaller class of nuclear groups, BB-reflexivity is equivalent to completeness.


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## Introduction

The character group $\Gamma G$ of an Abelian topological group $G$ is the set of all continuous homomorphisms from $G$ into the torus $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$, with pointwise multiplication. Homomorphisms from $G$ into $\mathbb{T}$ are usually named characters. The dual group of $G$ is defined as $\Gamma G$, endowed with the compact-open topology $\tau_{co}$. It will be denoted by $G^\wedge$,

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while $G^{\wedge\wedge} := (G^\wedge)^\wedge$ stands for the bidual. By $Hom(G, \mathbb{T})$ and $Hom(G, \mathbb{R})$ we denote homomorphisms of the corresponding groups, while $CHom(G, \mathbb{T})$ (also called $\Gamma G$) and $CHom(G, \mathbb{R})$ denote continuous homomorphisms.

The canonical embedding $\alpha_G : G \to G^{\wedge\wedge}$ is defined by $\alpha_G(g)(\chi) = \chi(g)$ for every $g \in G$ and every $\chi \in G^\wedge$. If $\alpha_G$ is a topological isomorphism, the topological group $G$ is called reflexive (more precisely, Pontryagin reflexive). The Pontryagin–Van Kampen theorem states that locally compact Abelian groups are reflexive. However the class of reflexive groups includes other types of groups, like complete metrizable locally convex spaces and reflexive topological vector spaces [22] (both classes considered as topological groups, i.e., forgetting the linear structure), arbitrary products of reflexive groups [17], complete metrizable nuclear groups [2], etc.

Our aim in this paper is to study completeness of a topological Abelian group and also of its dual, and how these properties are related with reflexivity. Since completeness of locally convex vector spaces is totally characterized by the Grothendieck theorem and its corollaries, it seems natural to center the question for locally quasi-convex groups and to start with the underlying group of a topological vector space. For such an object $E$, completeness is independent of the point of view, i.e., if it is looked at as a vector space or as a group. However the character group $\Gamma E$ is no longer a vector space, and is obviously different from the set of continuous linear forms $\mathcal{L}E$, which roughly speaking is the natural dual of a vector space. Thus, if a theorem of Grothendieck-type is to be obtained for the dual group of a locally convex vector space, some work must be done, even for this very particular class of topological groups.

On the other hand, the continuous convergence structure can be defined in the dual of a topological vector space and some fundamental results in duality theory heavily rely on it, although it may not be explicitly stated. Continuous convergence was first defined in the dual of a convergence group by Binz and Butzmann giving rise to the notion of BB-reflexive convergence groups [3]. In [8] it is proved that a locally convex vector space is BB-reflexive if and only if it is complete. In Corollary 4.4 we see that this result is also valid for nuclear groups, a class of topological Abelian groups introduced by Banaszczyk in [2].

## 1. Preliminary background

A topology defines in a natural way a convergence structure, namely, the one given by its convergent filters or nets. Conversely, one can start declaring which nets (or filters) on a set $X$ converge, and the corresponding limit points, and this is a convergence structure for the set $X$. If some general conditions (convergence axioms) are satisfied so that there exists a topology in $X$ for which the convergent nets (or filters) are the given “a priori” [18], it can be said that the convergence derives from a topology, or simply that it is topological.

If the convergence structure does not fulfill all the requirements to be derived from a topology, then we only have a convergence space. In the literature there is not an unanimous acceptance of which are the axioms that must define this concept. We are interested just on the continuous convergence structure and we have followed the text of Binz [3], where the reader can find a good account of information. We also take his notations. Topological notions such as continuity, cluster point, closed, open or compact sets, etc. can be stated in terms of convergence of filters or nets, therefore they have corresponding definitions for convergence spaces. Convergence groups are groups endowed with a convergence structure compatible with the group operation [15]. If $G$ is a convergence group, we also use the symbol $\Gamma G$ to denote the group of all continuous homomorphisms from $G$ into $\mathbb{T}$. The *continuous convergence structure* $\Lambda_c$ in $\Gamma G$ is defined in the following way: A filter $\mathcal{F}$ in $\Gamma G$ converges in $\Lambda_c$ to an element $\xi \in \Gamma G$ if for every $x \in G$ and every filter $\mathcal{H}$ in $G$ that converges to $x$, $e(\mathcal{F} \times \mathcal{H})$ converges to $\xi(x)$ in $\mathbb{T}$ (here, $e(\mathcal{F} \times \mathcal{H})$ denotes the filter generated by the sets $e(F \times H) := \{f(x);\ f \in F,\ x \in H\}$, where $F \in \mathcal{F},\ H \in \mathcal{H}$). By means of nets, the definition should be as follows: A net $(f_\alpha)_{\alpha \in D}$ in $\Gamma G$ is $\Lambda_c$-convergent to $f \in \Gamma G$ if for every net $(x_\beta)_{\beta \in E}$ in $G$ converging to $x \in G$, the net $(f_\alpha(x_\beta))_{(\alpha,\beta) \in D \times E}$ ($D \times E$ has the product direction) converges to $f(x)$ in $\mathbb{T}$.

It is well known that a topology in $\Gamma G$ for which the evaluation $e : \Gamma G \times G \to \mathbb{T}$ is continuous ($\Gamma G \times G$ has the natural product structure) must be finer than the compact open topology $\tau_{co}$, but $\tau_{co}$ itself very seldom makes $e$ continuous. Therefore a convergence structure may be designed in $\Gamma G$ in order to obtain the continuity of the evaluation mapping $e : \Gamma G \times G \to \mathbb{T}$ as well as the property of being the coarsest with this condition. This is the real motivation to introduce the continuous convergence structure on a dual. The dual group $\Gamma G$ of a convergence group $(G, \Lambda)$, endowed with the convergence structure $\Lambda_c$, is a convergence group, denoted by $\Gamma_c G$ and called the *convergence dual of* $G$.

A convergence group is called *BB-reflexive* if the canonical homomorphism $\kappa_G : G \to \Gamma_c \Gamma_c G$ is a bicontinuous isomorphism (here $\Gamma_c \Gamma_c G$ has the obvious meaning). Observe that, due to the continuity of $e : \Gamma_c G \times G \to \mathbb{T}$, $\kappa_G$ is always continuous. Analogously, a convergence vector space $E$ is *BB-reflexive as a space* if the canonical embedding $\iota_E : E \to \mathcal{L}_c \mathcal{L}_c E$ is a bicontinuous isomorphism. Here $\mathcal{L}_c E$ denotes the set of all continuous linear forms on $E$, endowed with the continuous convergence structure. In the category of Hausdorff topological groups, BB-duality and Pontryagin duality are independent notions [12], but they coincide, for instance, in the family of metrizable topological groups [11].

The compact open topology and the continuous convergence structure in the dual of a locally compact Abelian topological group, have the same convergent filters. This fact characterizes the locally compact groups in the class of reflexive topological groups [19].

If $E$ is a real topological vector space, the dual group $E^\wedge$, and the dual Pontryagin vector space $E^*$ (i.e., the set of all continuous linear forms endowed with the compact open topology) are related through the exponential mapping $f \to \exp(2\pi \mathrm{i} f)$, which in this case happens to be a topological isomorphism (see [2, (2.3)]). Here the compact open topology plays some role; it would not be a topological isomorphism if the supporting sets were endowed by the corresponding weak topologies.

The duality theory for topological vector spaces is usually restricted to locally convex spaces where the Hahn–Banach theorem holds. In an arbitrary topological group, the notion of convexity has no sense. Nevertheless, a similar notion, the so called quasi-convexity, was introduced by Vilenkin in [24], where he also defined the locally quasi-convex groups.

A subset $A$ of a topological group $G$ is called quasi-convex if for every $g \in G \setminus A$, there is some $\chi \in A^0 := \{\chi \in \Gamma G\colon \operatorname{Re}\chi(z) \geqslant 0,\ \forall z \in A\}$, such that $\operatorname{Re}\chi(g) < 0$. The quasi-convex hull of any subset $H \subset G$ is defined as the intersection of all quasi-convex subsets of $G$ containing $H$. An Abelian topological group $G$ is called locally quasi-convex if it has a neighborhood basis of the neutral element $e_G$, given by quasi-convex sets. The dual $G^\wedge$ of any topological Abelian group $G$ is locally quasi-convex. In fact, the sets $K^0$, where $K$ runs through the compact subsets of $G$, constitute a neighborhood basis of $e_{G^\wedge}$ for the compact open topology.

The additive group of a topological vector space is locally quasi-convex if and only if the vector space itself is locally convex [2]. Therefore it is natural to restrict the duality theory of topological Abelian groups to the locally quasi-convex ones. Some of the well known results on locally convex spaces have analogic versions valid for locally quasi-convex groups. In particular a topology on a group $G$ is locally quasi-convex if and only if it is an $\mathfrak{S}$-topology (uniform convergence topology) for the family $\mathfrak{S}$ of equicontinuous subsets of the dual $G^\wedge$ [13, Proposition 3.9]. A duality theory for groups is extensively presented in [13]. Here we will only state what is needed for our aims.

If $G$ is a topological group, the Bohr topology on $G$ is the weakest topology that makes continuous all characters of $\Gamma G$. We will denote it by $\omega(G, \Gamma G)$, and the pointwise topology on $\Gamma G$ will be denoted by $\omega(\Gamma G, G)$. Very interesting results on the Bohr topology of a locally compact Abelian group, from a topological point of view, are obtained in [14].

The paper is organized as follows: in Section 2 we present examples of complete metrizable locally quasi-convex groups which are not Pontryagin reflexive. In doing so we are concerned with lifting of characters on a group $G$ to homomorphisms from $G$ into $\mathbb{R}$. We use essentially a result of Nickolas.

In Section 3 we present the Grothendieck completeness theorem for the underlying group of a locally convex space and its dual group.

In the last section we see that the most natural version of the Grothendieck theorem for topological groups does not hold, even in the class of metrizable locally quasi-convex groups. The examples which prove it, are precisely the groups considered in Section 2. We then study a weaker form of the Grothendieck theorem valid for locally quasi-convex groups and prove that for the smaller class of nuclear groups, or that of locally convex vector groups, the result can be improved.

## 2. A family of nonreflexive complete metrizable locally quasi-convex groups

The groups $L^p_{\mathbb{Z}}[0,1]$, for $1 \leqslant p < \infty$, have the properties mentioned in the title of this section. For the sake of completeness we describe here these groups.

Let $L^p[0,1]$ or simply $L^p$ be the vector space of all classes of real measurable functions $f$ such that $\|f\| := (\int_0^1 |f(t)|^p \mathrm{d}t)^{1/p} < \infty$. It is well known that the spaces $L^p$ endowed with the norm $\|\ \|$ are Banach spaces. Now $L^p_{\mathbb{Z}}$ is the subset of $L^p$ of all the classes of integer valued functions, with the induced topology. Evidently it is a complete metrizable locally quasi-convex topological Abelian group, but it is not a vector subspace.

Now we summarize the steps which lead to the proof that $L^p_{\mathbb{Z}}$ is a nonreflexive group. Crucial to all of them is the following result of Nickolas:

$$\text{If an Abelian topological group } G \text{ is a k-space, then the path component of the identity in } G^\wedge \text{ is the union of all the one-parameter subgroups of } G^\wedge. \qquad (*)$$

By a one-parameter subgroup of $G$ it is commonly understood the image of $\mathbb{R}$ by a continuous homomorphism from $\mathbb{R}$ into $G$.

First we are concerned with lifting of characters to real valued characters. As we already mentioned, every continuous character defined in a topological vector space can be lifted to a continuous linear form. The same assertion can be made for certain groups, as we expose in the next proposition. Its proof is essentially contained in the proof of $(*)$, given in [20].

**Proposition 2.1.** *Let $G$ be a topological Abelian group such that $G$ is a k-space and its dual $G^\wedge$ is pathwise connected. Then every continuous homomorphism $\varphi : G \to \mathbb{T}$ can be lifted to a continuous homomorphism $\tilde{\varphi} : G \to \mathbb{R}$ such that $p\tilde{\varphi} = \varphi$, where $p : \mathbb{R} \to \mathbb{T}$ is the covering projection.*

**Remark.** The assumption that $G$ is a k-space is not a necessary condition. In [1, Corollary 8.12], an example of a locally convex vector space $E$, which is not a k-space is presented. Clearly, the lifting property for $E$ derives from the natural isomorphism between $E^*$ and $E^\wedge$.

We can now state the following:

**Theorem 2.2.** *If $G$ is a metrizable, reflexive pathwise connected Abelian group, then*:

(a) *Every continuous character $\varphi : G^\wedge \to \mathbb{T}$ can be lifted to a real continuous character (i.e., there exists $\tilde{\varphi} : \in CHom(G^\wedge, \mathbb{R})$ such that $p\tilde{\varphi} = \varphi$),*

(b) *$G$ is the union of its one-parameter subgroups,*

(c) *$G$ is divisible.*

**Proof.** (a) If $G$ is metrizable, $G^\wedge$ is a k-space as shown in [11]. On the other hand $(G^\wedge)^\wedge$ is topologically isomorphic to $G$, therefore pathwise connected. By Proposition 2.1, every continuous character $\varphi : G^\wedge \to \mathbb{T}$ can be lifted to say $\tilde{\varphi} : G^\wedge \to \mathbb{R}$ such that $p\tilde{\varphi} = \varphi$. Furthermore the lifting is unique (see [23, p. 69, 2nd paragraph]), since any lifting to a continuous character $\tilde{\tilde{\varphi}}$ must be such that $\tilde{\tilde{\varphi}}(\nu_0) = 0 \in \mathbb{R}$, where $\nu_0$ is the neutral element of $G^\wedge$.

(b) Follows also from $(*)$.

(c) In order to prove the last assertion, we express $G$ as the union of its one-parameter subgroups, say $G = \bigcup\{\xi(\mathbb{R}) \colon \xi \in CHom(\mathbb{R}, G)\}$. For any $x \in G$ and any $n \in \mathbb{N}$, there exists $\xi \in CHom(\mathbb{R}, G)$ and $r \in \mathbb{R}$, such that $\xi(r) = x$. Now $\xi(r/n)$ is such that $n\xi(r/n) = x$.

**Remark.** A topological group which is the union of its one-parameter subgroups must be pathwise connected. Thus, the condition that $G^\wedge$ be pathwise connected cannot be dropped in Proposition 2.1.

It was known to Dixmier (see [16, p. 393]) that for a locally compact Abelian group $G$ the condition that every character in $G$ can be lifted to a real character is equivalent to the fact that the dual $G^\wedge$ is the union of its one-parameter subgroups. That this also holds for metrizable reflexive groups can be deduced from the proof of $(*)$ together with Theorem 2.2.

**Proposition 2.3.** *The group* $G = L^p_{\mathbb{Z}}[0,1]$ $(p \geqslant 1)$ *is not Pontryagin reflexive.*

**Proof.** The proof follows easily from the fact that $G$ is contractible, therefore pathwise connected. Since it is a metrizable group, if it were reflexive, $G$ would satisfy all the assumptions of Theorem 2.2, therefore it would be divisible. But this is not the case; obviously for the function $f$ constant to one, there is no $g \in L^p_{\mathbb{Z}}$ such that $2g = f$. The fact that $L^p_{\mathbb{Z}}$ is contractible can be seen in [1]. Nevertheless we sketch the proof. Denote by $\chi_{[0,l)}$ the characteristic function of $[0,l)$ in $[0,1]$. The mapping

$$\begin{aligned} F : L^p_{\mathbb{Z}} \times [0,1] &\to L^p_{\mathbb{Z}} \\ (f,t) &\to \chi_{[0,1-t)} \cdot f \end{aligned}$$

establishes a homotopy between the identity mapping in $L^p_{\mathbb{Z}}$ and the constant to null mapping. It is therefore a contraction of $L^p_{\mathbb{Z}}$. □

**Remark.** The fact that $L^p_{\mathbb{Z}}$ is not Pontryagin reflexive had been proved earlier by Aussenhofer in [1]. She follows a different method and gives also a precise description of the dual of $L^p_{\mathbb{Z}}$.

## 3. The Grothendieck completeness theorem on the additive group of a locally convex vector space

In this section we prove that the underlying group of a topological vector space and its dual group satisfy an analogue to the Grothendieck theorem (GT). We first give a few lemmas which will simplify our job.

In the next propositions, $E$ will denote a topological vector space. We keep the standard notations $E^\wedge$, $\Gamma_c E$ and $\Gamma_{\mathfrak{S}} E$ for the character group of $E$, endowed with the compact open topology, with the continuous convergence structure and with an $\mathfrak{S}$-topology, respectively. Also by $E^*$, by $\mathcal{L}_c E$ and by $\mathcal{L}_{\mathfrak{S}} E$ we mean the set of continuous linear forms $\mathcal{L}E$ endowed with the compact open topology, with the continuous convergence structure and with an $\mathfrak{S}$-topology, respectively.

**Lemma 3.1.** *Let $(E,\tau)$ be a locally convex vector space and let $\mathfrak{S}$ be a family of closed bounded and balanced sets covering $E$.*

(i) *Denote by $\rho : Lin(E,\mathbb{R}) \to Hom(E,\mathbb{T})$ the exponential mapping, $\rho(f) = \exp(2\pi \mathrm{i} f)$, $\forall f \in Lin(E,\mathbb{R})$. A character $\varphi$ belongs to $\mathrm{Im}(\rho)$ if and only if $\varphi|_L$ is continuous for all one-dimensional vector subspace $L \subset E$.*

(ii) *The following assertions are equivalent*:

  (a) *Every character with continuous restriction on all $S \in \mathfrak{S}$, is continuous.*

  (b) *Every linear form with continuous restriction on all $S \in \mathfrak{S}$, is continuous.*

**Proof.** (i) Suppose $\varphi = \rho(f)$ for some $f \in Lin(E,\mathbb{R})$. If $L \subset E$ is a one-dimensional vector subspace, $f|_L$ is continuous, therefore $\varphi|_L$ is continuous.

Conversely, let $\varphi \in Hom(E,\mathbb{T})$. Denote by $[a]$ the subspace generated by a non null vector $a \in E$. Since $\varphi|_{[a]}$ is continuous, it can be considered as a continuous character defined on $\mathbb{R}$, and consequently there is a unique real number $t_a$ such that $\varphi(ra) = \exp(2\pi \mathrm{i} t_a r)$ for all $r \in \mathbb{R}$. It is easy to check that $t_{\lambda a} = \lambda t_a$ and $t_{a+b} = t_a + t_b$, for any $\lambda \in \mathbb{R}$ and any $b \in E$. Therefore by superposition of the one-dimensional linear forms $f_a(ra) = t_a r$ we obtain a linear form $f : E \to \mathbb{R}$. Clearly $\varphi = \exp(2\pi \mathrm{i} f)$.

(ii) (a) $\Rightarrow$ (b) Let $f : E \to \mathbb{R}$ be a linear form continuous on all $S \in \mathfrak{S}$. The corresponding character $\exp(2\pi \mathrm{i} f)$ is continuous in all $S \in \mathfrak{S}$, and by (a) it is continuous. Therefore, by [2, (2.3)], $f$ is continuous.

(b) $\Rightarrow$ (a) Let $\chi : E \to \mathbb{T}$ be a character with continuous restriction on each $S \in \mathfrak{S}$. From this it is easily seen that the restriction of $\chi$ to finite-dimensional subspaces is continuous and, by (i), there exists a linear form $f : E \to \mathbb{R}$ such that $\exp(2\pi \mathrm{i} f) = \chi$.

Now for $S \in \mathfrak{S}$, and $\varepsilon > 0$, there is some balanced neighborhood of $e$ such that $|\exp(2\pi \mathrm{i} f(x)) - 1| < \varepsilon/2$, for all $x \in S \cap U$. Then $|\exp(2\pi \mathrm{i} t f(x)) - 1| < \varepsilon/2$, for all $|t| \leqslant 1$, and all $x \in S \cap U$. Consequently $|f(x)| < \varepsilon$ and the restriction of $f$ to all elements of $\mathfrak{S}$ is continuous. By (b) $f$ is continuous on $E$, and so is $\chi = \exp(2\pi \mathrm{i} f)$. □

**Lemma 3.2.** *Let $(E,\tau)$ be a locally convex vector space and let $\mathfrak{S}$ be a family of closed bounded and balanced sets covering $E$. The exponential mapping $\rho : \mathcal{L}_{\mathfrak{S}} E \to \Gamma_{\mathfrak{S}} E$ is a topological isomorphism.*

**Proof.** The continuity of $\rho$ is straightforward, and holds without any conditions on the sets $S \in \mathfrak{S}$. An argument similar to that of (b) $\Rightarrow$ (a) of the previous lemma proves the continuity of the inverse mapping. In fact only the properties that the sets $S \in \mathfrak{S}$ are balanced and cover $E$ are used. □

Next we state that the continuous convergence restricted to equicontinuous subsets of $\Gamma G$ coincides with the pointwise convergence. The proof is straightforward.

**Lemma 3.3.** *Let $G$ be a topological group and let $H$ be an equicontinuous subset of $\Gamma G$. If $(\xi_\alpha)$ is a net contained in $H$ and $\xi \in \Gamma G$, the following assertions are equivalent*:

(1) *$(\xi_\alpha)$ is $\Lambda_c$-convergent to $\xi$.*

(2) $(\xi_\alpha)$ *is* $\tau_{co}$*-convergent to* $\xi$.
(3) $(\xi_\alpha)$ *is* $\omega(\Gamma G, G)$*-convergent to* $\xi$.

By the previous lemma equicontinuous subsets of $\Gamma_c G$ are topological. The family of closed equicontinuous subsets of $\Gamma G$ actually coincides with that of $\Lambda_c$-compact subsets. If $\alpha_G$ is continuous, then they also coincide with the family of $\tau_{co}$-compact subsets. For complete metrizable groups, more can be said. The following statement is comparable to the uniform boundedness principle. Since the latter is a significant result in the theory of topological vector spaces, one can reasonably expect that this sort of "equicontinuity principle" may have some importance for Abelian topological groups. The proof of it can be seen in [13, Theorem 1.5], where it is established in a more general setting.

**Lemma 3.4.** *If* $G$ *is a complete metrizable topological Abelian group, then every* $\omega(\Gamma G, G)$*-compact subset of* $\Gamma G$ *is equicontinuous.*

The convergence dual of a topological group is locally compact, and has properties similar to those of k-spaces.

**Lemma 3.5.** *Let* $G$ *be a topological group. The following assertions hold*:
(1) $\Gamma_c G$ *is a locally compact convergence group.*
(2) *If a character* $\chi : \Gamma_c G \to \mathbb{T}$ *is such that* $\chi|_K$ *is continuous for all compact* $K \subset \Gamma_c G$, *then* $\chi$ *is continuous.*
(3) $\Gamma_c \Gamma_c G$ *is topological and carries the compact open topology relative to the compact subsets of* $\Gamma_c G$. *Furthermore, it is complete.*

**Proof.** It can be seen in [6, (3.2.2), (1.5.4) and (3.2.5)]. □

**Lemma 3.6.** *If* $G$ *is a Hausdorff locally quasi-convex group, then* $\kappa_G : G \to \kappa_G(G) \subset \Gamma_c \Gamma_c G$ *is an embedding.*

**Proof.** In order to prove that $\kappa_G$ is open and injective, take into account Lemma 3.5(3) and follow the proof of the same facts for $\alpha_G$ [2, (14.3)]. On the other hand $\kappa_G$ is always continuous. □

Next we see that $\omega(G, \Gamma G)$, and $\omega(\Gamma G, G)$ are the natural analogs to the weak and to the weak* topologies defined in a topological vector space and in its dual respectively.

**Lemma 3.7.** *Let* $G$ *be an Abelian topological group.*
(1) *The dual group of* $(G, \omega(G, \Gamma G))$ *is* $\Gamma G$.
(2) *If* $\Gamma G$ *separates points of* $G$, *then every continuous character on* $(\Gamma G, \omega(\Gamma G, G))$ *is an evaluation at some point of* $G$, *i.e., the dual group of* $(\Gamma G, \omega(\Gamma G, G))$ *can be algebraically identified with* $G$.

**Proof.** It can be seen in [13, Theorem 3.7]. □

The identification of Lemma 3.7(2) is even topological for some classes of groups, as we prove now:

**Theorem 3.8.** *If $G$ is a complete metrizable locally quasi-convex group, then the dual group $X = (\varGamma G, \omega(\varGamma G, G))^\wedge$ is topologically isomorphic to $G$.*

**Proof.** By Lemma 3.7(2), $X$ can be algebraically identified with $G$. If $K \subset \varGamma G$ is $\omega(\varGamma G, G)$-compact, then it is equicontinuous by Lemma 3.4. This means that ${}^0K := \{z \in G\colon \operatorname{Re}\chi(z) \geqslant 0, \forall \chi \in K\}$ is a 0-neighborhood in $G$. On the other hand ${}^0K$ can be identified with $K^0$, and so we have that every 0-neighborhood in $X$ is a 0-neighborhood in $G$.

Conversely, if $V$ is a quasi-convex 0-neighborhood in $G$, $V^0$ is $\omega(\varGamma G, G)$-compact [2, 1.5], therefore $V = {}^0(V^0)$ is a 0-neighborhood in $X$. □

**Corollary.** *The Pontryagin dual of a topological Abelian group is not necessarily reflexive.*

**Proof.** Take $E := (\varGamma G, w(\varGamma G, G))$, with $G = L^p_{\mathbb{Z}}[0, 1]$, being $p > 1$, and follow the argument of Theorem 3.8.

For a locally convex vector space $E$, we bring together the two view points, as a group and as a space, in the next two theorems.

**Theorem 3.9.** *Let $(E, \tau)$ be a locally convex space and let $\mathfrak{S}$ be a family of closed bounded convex and balanced sets covering $E$. The following statements are equivalent*:

(a) *$\mathcal{L}E$ is complete under the $\mathfrak{S}$-topology.*
(b) *Every linear form $f$ on $E$ which is $\tau$-continuous on each $S \in \mathfrak{S}$, is continuous on $(E, \tau)$.*
(c) *$(\mathcal{L}E, \tau_{\mathfrak{S}})$ is BB-reflexive, i.e., $E$ is bicontinuously isomorphic to $\mathcal{L}_c\mathcal{L}_c(\mathcal{L}E, \tau_{\mathfrak{S}})$ .*
(d) *The group $(\varGamma E, \tau_{\mathfrak{S}})$ is complete.*
(e) *Every character on $E$, which is continuous on each $S \in \mathfrak{S}$, is continuous on $(E, \tau)$.*
(f) *$(\varGamma E, \tau_{\mathfrak{S}})$ is BB-reflexive, i.e., it is bicontinuously isomorphic to $\varGamma_c\varGamma_c(\varGamma E, \tau_{\mathfrak{S}})$ .*

**Proof.** The equivalence between (a) and (b) is exactly the Grothendieck theorem. The proof can be seen in any classical treatise, for example, [21]. In [8] it is proved that a locally convex vector space is complete if and only if it is BB-reflexive as a vector space, thus (a) $\Leftrightarrow$ (c).

(b) $\Leftrightarrow$ (e) is precisely (ii) of Lemma 3.1.

(c) $\Leftrightarrow$ (f) and (a) $\Leftrightarrow$ (d) are obtained through the topological isomorfism $\rho : \mathcal{L}_{\mathfrak{S}}E \to \varGamma_{\mathfrak{S}}E$ (Lemma 3.2). □

**Remark.** For any topological vector space $E$, $\mathcal{L}_cE$ is BB-reflexive, without any conditions on $E$ [4]. Taking into account that $\mathcal{L}_cE$ is bicontinuously isomorphic to $\varGamma_cE$ (for any convergence vector space $E$, Satz 1 of [9]) it can be easily proved that also $\varGamma_cE$ is BB-reflexive as a group.

**Theorem 3.10.** *Let $(E, \tau)$ be a Hausdorff locally convex space. The following assertions are equivalent*:

(a) *$E$ is complete.*

(b) *Every linear form on $\mathcal{L}E$ which is $\omega(\mathcal{L}E, E)$-continuous on every equicontinuous subset of $\mathcal{L}E$, is $\omega(\mathcal{L}E, E)$-continuous on all of $\mathcal{L}E$.*

(c) *Every character on $\Gamma E$ which is $\omega(\Gamma E, E)$-continuous on every equicontinuous subset of $\Gamma E$, is $\omega(\Gamma E, E)$-continuous on all $\Gamma E$.*

(d) *$E$ is BB-reflexive as a topological vector space.*

(e) *$E$ is BB-reflexive as a topological group.*

**Proof.** (a) $\Leftrightarrow$ (b) is a standard corollary of GT, see, for example, [21].

(a) $\Leftrightarrow$ (d) and (d) $\Leftrightarrow$ (e) are proved in [8] and [9], respectively.

(c) $\Rightarrow$ (e) In order to see that $\kappa_E$ is a topological isomorphism, only surjectivity is to be seen, since $\kappa_E$ is already an embedding (Lemma 3.6). Let $\chi \in \Gamma\Gamma_c E$. If $H \subset \Gamma E$ is equicontinuous, $\chi|_H$ is $\omega(\Gamma E, E)$-continuous by Lemma 3.3. We apply (c) together with Lemma 3.7(2) and we obtain that there is some $x \in E$ such that $\chi = \kappa_E(x)$.

(e) $\Rightarrow$ (c) Let $\chi : \Gamma E \to \mathbb{T}$ be a character such that $\chi|_H$ is $\omega(\Gamma E, E)$-continuous for all $H \subset \Gamma E$ equicontinuous. Taking into account that $\Gamma_c E$ is a locally compact convergence group and that every compact subset of $\Gamma_c E$ is equicontinuous, by Lemma 3.5(3) we have that $\chi$ is a $\Lambda_c$-continuous character. Applying now (e) there exists $x \in E$ such that $\kappa_G(x) = \chi$. Thus $\chi$ is $\omega(\Gamma E, E)$-continuous. $\square$

In Theorem 3.10 BB-reflexivity cannot be substituted by reflexivity in ordinary sense. There is a famous example of Komura of a noncomplete locally convex vector space $E$ which is topologically isomorphic to $(E_b^*)_b^*$. Here $E_b^*$ denotes the dual vector space endowed with the topology of uniform convergence on the weakly bounded subsets of $E$.

## 4. The Grothendieck theorem for locally quasi-convex groups

In this section we deal with some approximation to the Grothendieck theorem, for Hausdorff locally quasi-convex groups. Comparing Theorems 3.9 and 3.10 with the results obtained in this section, we see that, with respect to completeness, the underlying groups of topological vector spaces behave better than locally quasi-convex groups in general. The equivalence between (a) and (b) in Theorem 4.1 confirms in a sense that the tools of continuous convergence and BB-duality theory are appropriate in order to obtain a generalization of the Grothendieck theorem.

**Theorem 4.1.** *Let $G$ be a Hausdorff locally quasi-convex topological group. Consider the statements*:

(a) *$G$ is BB-reflexive.*

(b) *Every character on $\Gamma G$ which is $\omega(\Gamma G, G)$-continuous on every equicontinuous subset of $\Gamma G$, is $\omega(\Gamma G, G)$-continuous on all of $\Gamma G$.*

(c) *$G$ is complete and $\alpha_G$ is surjective.*
*Then* (a) *is equivalent to* (b) *and they imply* (c).

**Proof.** For the proof of (a) ⇔ (b), mimic the proof of (b) ⇔ (e) in Theorem 3.10, since the vector space structure is not used there.

We now prove that each of them implies (c). Let us show that $\alpha_G$ is surjective. Take any continuous character $\psi$ on $G^\wedge$. Since on equicontinuous subsets of $G^\wedge$ the compact open topology coincides with the pointwise one, it follows that $\psi$ satisfies the assumption of (b), thus it is pointwise continuous on $G^\wedge$ and so, by Lemma 3.7(2), it belongs to $\alpha_G(G)$.

In order to prove that $G$ is complete, we use the following general theorem [5, Chapter X, Section 6, Corollary 2 to Theorem 2]:

> *Let $X$ be a topological space, $\mathfrak{S}$ a collection of subsets of $X$ and $Y$ a complete uniform space. Then, the space of all maps from $X$ into $Y$ whose restrictions to the sets of $\mathfrak{S}$ are continuous, equipped with the topology of uniform convergence on the sets of $\mathfrak{S}$, is complete.*

Take $X$ as $\Gamma_{co}G$, $Y = \mathbb{T}$ and $\mathfrak{S}$ as the family of all equicontinuous subsets of $\Gamma G$. By the quoted theorem, the space $\mathcal{H}(X, \mathbb{T})$ of all maps whose restriction to the sets of $\mathfrak{S}$ are continuous, endowed with the $\mathfrak{S}$-topology, is complete. Observing that $\mathfrak{S}$ covers $\Gamma G$, we obtain that the subset formed by all characters in $\mathcal{H}(X, \mathbb{T})$ is closed, therefore also complete. By (b) together with Lemma 3.5 we have that the latter coincides with the set of all characters $\omega(\Gamma G, G)$-continuous, and by Lemma 3.7 it can be algebraically identified with $G$. On the other hand, taking into account that $G$ is locally quasi-convex, its original topology coincides with the $\mathfrak{S}$-topology, where $\mathfrak{S}$ is the family af all equicontinuous subsets of $\Gamma G$ (see [13, Proposition 3.9]). Thus the identification is also topological and $G$ is complete. □

Observe that for locally convex vector spaces completeness is equivalent to BB-reflexivity ((a) ⇔ (d) of Theorem 3.10). However, an analog in the framework of locally quasi-convex groups does not hold as we now state:

**Corollary 4.2.** *Let $G$ be a Hausdorff locally quasi-convex group. The implications* (a) ⇒ (c) *and* (a) ⇒ (e) *of Theorem* 3.10 *do not hold even if $G$ is complete metrizable and separable.*

**Proof.** Take $G := L^p_{\mathbb{Z}}[0,1]$ (see Section 2). Being $G$ a closed subgroup of $L^p[0,1]$, it is complete. Since $G$ is metrizable and locally quasi-convex, $\alpha_G$ is continuous, injective and open in its image. By Proposition 2.3 $G$ is not reflexive, therefore $\alpha_G$ is not surjective. Now we apply Theorem 4.1. □

**Corollary 4.3.** *Let $G$ be a Hausdorff locally quasi-convex group. The implication* (d) ⇒ (e) *of Theorem* 3.9 *does not hold even if $G$ is a $\sigma$-compact hemi-compact locally quasi-convex group with the property that the quasi-convex hull of any compact subset is again compact.*

**Proof.** Take $G$ as in the previous corollary and put $E = (\Gamma G, \omega(\Gamma G, G))$. The $\mathfrak{S}$-family will be now the set of all $\omega(\Gamma G, G)$-compact subsets of $\Gamma G$. The group $\Gamma_{\mathfrak{S}} E$ is precisely $E^{\wedge}$ and by Theorem 3.8 can be identified with $G$, therefore it is complete. As proved in Corollary 4.2 $\alpha_G$ is not surjective. Now we apply Theorem 4.1. □

Each of the properties mentioned in Theorem 4.1(c), separately, do not imply (a) or (b) as shown by the groups $G = L^p_{\mathbb{Z}}[0, 1]$ and the Komura space respectively. We do not know if (c) implies (a) and (b). For the very special class of nuclear groups [2], the following can be stated:

**Corollary 4.4.** *Let $G$ be a nuclear topological group (or a locally convex vector group). The following assertions are equivalent:*

(a) *$G$ is BB-reflexive.*
(b) *Every character on $\Gamma G$ which is pointwise continuous on every equicontinuous subset of $\Gamma G$, is pointwise continuous on all of $\Gamma G$.*
(c) *$G$ is complete.*

**Proof.** We prove that completeness is equivalent to BB-reflexivity, and the equivalence between (a) and (b) is as in Theorem 4.1.

Let $G$ be a complete nuclear topological group. By [1, Theorem 21.3], $G$ can be embedded as a dually closed and dually embedded subgroup of a product of complete, metrizable nuclear groups. The BB-reflexivity of $G$ is proved taking into account the following facts:

(1) Every complete metrizable nuclear group is reflexive in Pontryagin sense [2, (17.3)].
(2) Metrizable Pontryagin reflexive groups are BB-reflexive [11]. Thus every factor group in the above mentioned product is BB-reflexive.
(3) Products of BB-reflexive groups are BB-reflexive [10].
(4) Dually closed and embedded subgroups of BB-reflexive groups are also BB-reflexive [7].

Conversely, any BB-reflexive group must be complete. The proof for locally convex vector groups is similar [2, (15.7)]. □

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