DYNAMICS OF TWO LOGISTIC MAPS WITH A MULTIPLICATIVE COUPLING

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Received October 16, 1991; Revised February 4, 1992

The dynamical behavior of a system formed by two symmetrically coupled logistic maps with a multiplicative coupling is analyzed. The transition to chaos (Ruelle–Takens type) and the multifractal properties of the attractor have been determined in this system. This transition cannot be deduced from the subharmonic cascade typical of a single logistic map. Under the same kind of symmetry, different classes of coupling in 2D maps give the same qualitative route to chaos but with different geometrical transition mechanisms.

1. Introduction

In the last years many systems have been studied to test new definitions, methods and ideas on complexity. Cellular Automata, coupled logistic and circle maps lattices are usual examples used to study properties of complex behavior. Some Cellular Automata show complexity even for simple rules [Wolfram, 1986]. As they are discrete in space and time they can be easily generated on a computer. However, the results are not easily interpretable in terms of dynamical systems. Slightly more complex (not discrete in local space) are coupled map lattices [Crutchfield & Kaneko, 1987] that can show different kind of "phases" and behaviors. The advantage of this systems is that the basic elements are well known and one can try to understand the "emergent" collective properties through some "average" procedure on the components [Kaneko, 1985].

As stressed in recent papers, the behavior of complex spatially extended systems cannot be covered by a simple diffusive coupling among oscillators. Therefore, some authors have proposed analyzing a global coupling with a "mean field" interaction [Kaneko, 1989]. A different behavior will be obtained with a multiplicative coupling.

As a first step, one can study two coupled logistic maps. In some recent works the dynamics of these maps under different couplings have been analyzed. For 2D maps with a reflection symmetry with respect to the diagonal, the Ruelle–Takens route (fixed point, period-two orbit, two limit cycles, chaos) is often obtained for both additive and multiplicative couplings. Usually the transition to chaos is via the broadening of the two limit cycles that become chaotic [Yuan *et al.*, 1983; Hogg & Huberman, 1984; Schult *et al.*, 1987].

We propose a kind of multiplicative coupling whose dynamics shows a bifurcation diagram that leads to the Ruelle–Takens route, but with a mechanism different from that in the 2D maps referenced above.

2. Multiplicatively Coupled Logistic Map

2.1. *The bimap*

In the present paper, we consider a logistic map whose parameter is not fixed but itself follows a logistic dynamics. In order to have a nontrivial dynamics, one forces the parameter to remain in the interval [1, 4]. Then the parable always has a fixed point different from zero that ensures a self-sustained dynamics.

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Therefore we take two logistic maps

$$x_{n+1} = \mu(y_n)\, x_n(1 - x_n) \quad y_{n+1} = \mu'(x_n)\, y_n(1 - y_n) \tag{1}$$

where $\mu(y_n)$ and $\mu'(x_n)$ are in the interval [1, 4]. The simplest choice that satisfies these conditions is the linear one

$$\mu = b(3y_n + 1), \quad \mu' = b(3x_n + 1) \ . \tag{2}$$

We add an adjustable parameter b in order to have different dynamical evolutions. Therefore, we arrive at a map with a natural reflection symmetry, that takes the form

$$x_{n+1} = f_b(x_n, y_n) \quad y_{n+1} = f_b(y_n, x_n) \tag{3}$$

where

$$f_b(x_n, y_n) = b(3y_n + 1)\, x_n(1 - x_n) \ . \tag{4}$$

We will refer to it as the **bimap** in the following.

2.2. *The stability of the bimap*

For the sake of convenience we will distinguish two groups of fixed points of the bimap

A) on the axes

$$p_0 = (0, 0), \quad p_1 = \left(\frac{b-1}{b}, 0\right), \quad p_2 = \left(0, \frac{b-1}{b}\right) \ . \tag{5}$$

B) on the diagonal

$$p_{3,4} = \frac{1}{3}\left\{1 \mp \left(4 - \frac{3}{b}\right)^{1/2}, 1 \mp \left(4 - \frac{3}{b}\right)^{1/2}\right\} \ . \tag{6}$$

We analyze separately the stability of these two groups of points.

Group A. i) For $0 < b < 1$, p_0 is a sink and $p_{1,2}$ are hyperbolic points on the negative side of the axes. ii) When $b = 1$, $p_0 = p_1 = p_2$. iii) For $b > 1$, p_0 is a source point and $p_{1,2}$ are hyperbolic points. Now the stable manifold of p_1 is the positive side of the x-axis and the stable manifold of p_2 is the positive side of the y-axis.

Group B. The stability of the points $p_{3,4}$ on the diagonal leads to some interesting situations. i) For $0 < b < 3/4$, the points $p_{3,4}$ are not possible solutions. ii) For $b = 3/4$, $p_3 = p_4$ is a stable point. iii) For $3/4 < b < \sqrt{3}/2$, p_3 is an hyperbolic point. Its unstable direction coincides with the stable manifold of p_4 (This point is a sink in this interval). Therefore, the diagonal between p_3 and p_4 forms a heteroclinic orbit. iv) When $b > \sqrt{3}/2$, p_4 destabilizes via a Pitchfork bifurcation, becoming a hyperbolic point that generates a stable period-two orbit formed by points $p_{5,6}$

$$p_{5,6} = \left(\frac{2b(b+1) \mp \sqrt{b(b+1)(4b^2-3)}}{b(4b+3)}, \frac{2b(b+1) \pm \sqrt{b(b+1)(4b^2-3)}}{b(4b+3)}\right) \ . \tag{7}$$

These have been calculated using the reflection symmetry of the attractor. They always lie on a line parallel to the transversal diagonal. v) The period-two orbit losses stability via a Hopf bifurcation: each point of this orbit gives rise to a limit cycle for $b = 0.957$. However, the iterations continue to alternate between the two limit cycles and then are characterized mainly by two frequencies (quasiperiodic). These limit cycles grow when b increases further and, for some small intervals of b, frequency locking windows are obtained.

For b slightly larger than 1, the limit cycles approach the stable manifold of the hyperbolic point p_4 giving rise to a folding process. However, the system is still quasiperiodic. When b reaches the value $b = 1.029$, the limit cycles can cross the stable manifold of p_4 that coincides with the heteroclinic orbit between p_0 and p_4 and some irregular motions appear around this point. For $1.029 < b < 1.0843$ the limit cycles still grow and fold, becoming very complex. When the limit value $b = 1.084322$ is reached, the attractor is tangent to its basin boundary and the iterates can cross the axes. They are attracted by the stable manifold (the axes) of the saddle points $p_{1,2}$, but when they arrive at the neighborhood of the unstable manifolds of these points, they can escape to infinity and the attractor disappears.

We represent in Fig. 1 the evolution of the iterates for different values of b. These figures have been obtained by plotting 1.5×10^4 iterations, starting from an arbitrary initial point. Figure 1a shows the iteration between the two points p_5 and p_6 ($b = 0.900$). In Fig. 1b the spiraling of the iterates around $p_{5,6}$ for $b = 0.956$ is clearly apparent. These spirals preclude the formation of the limit cycles that are shown in Fig. 1c. ($b = 1$). The folding of these cycles around p_4 appears for $b = 1.03$ in Fig. 1d. A typical chaotic situation

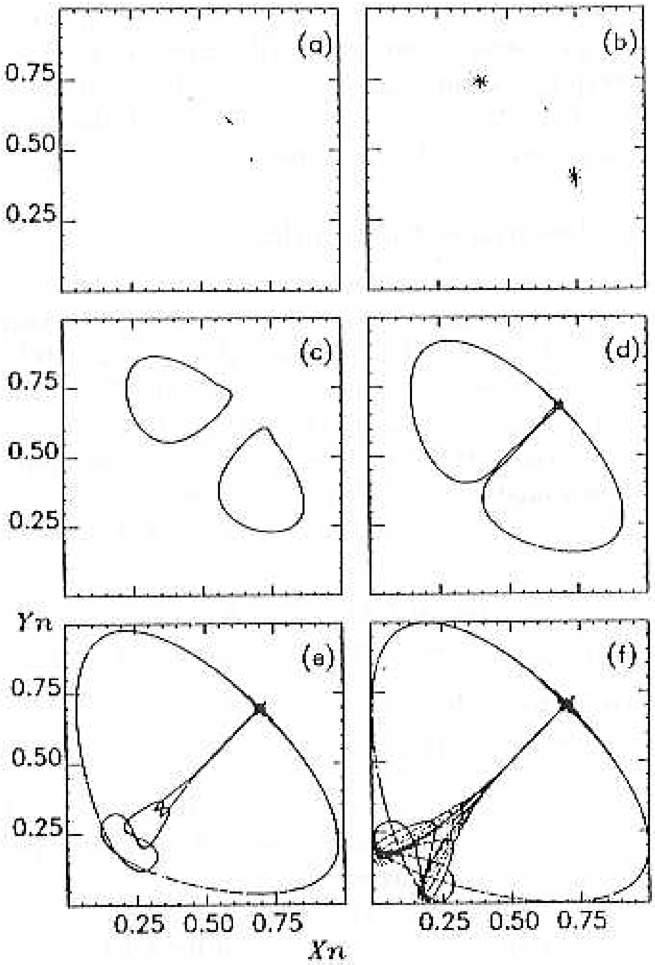

Fig. 1. Iterates of the bimap for different values of the parameter b: (a) $b = 0.900$, (b) $b = 0.956$, (c) $b = 1.000$, (d) $b = 1.030$, (e) $b = 1.070$, (f) $b = 1.0834$.

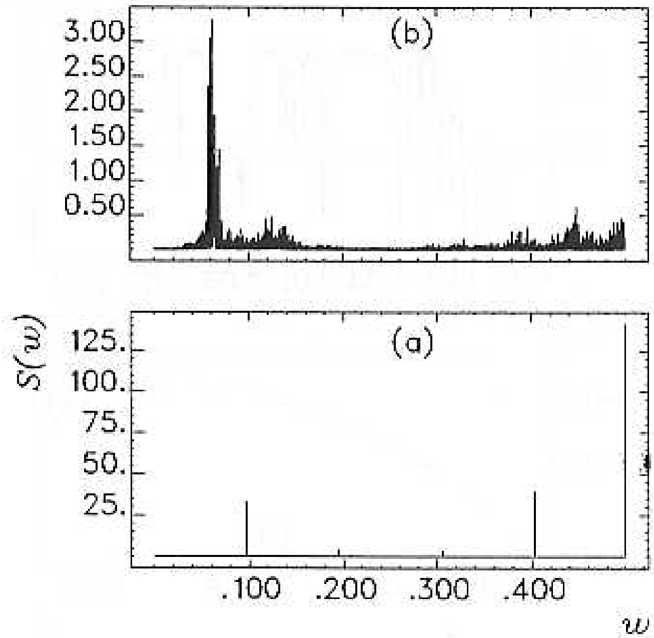

Fig. 2. Fourier spectra $S(w)$ (arbitrary units) of the iterates as a function of the frequency w for two representative values of b. (a) Stable limit cycles ($b = 1.020$). (b) Chaotic attractor, ($b = 1.070$).

($b = 1.070$) is given in Fig. 1e. The iterations in this range of b give rise to an interlacing of the two limit cycles on the diagonal and a complex folding process around the unstable upper fixed point p_4. The complexity of the iterates is always localized around two regions: near p_4 and near the hyperbolic points $p_{1,2}$. Finally for the limit value $b = 1.0834$ the complex attractor is given in Fig. 1f.

2.3. *Metric and statistical measures*

The power spectrum $S(w)$ of this attractor as a function of the frequency w for two typical values of b is represented in Fig. 2. In the region where the two limit cycles are stable, one observes different sharp peaks: one of high frequency ($w_1 = 0.5$) comes from the alternation between the two limit cycles in the iteration process; a second one w_2 (which is b-dependent) is associated with the characteristic frequency of each cycle. The rest of the peaks correspond to the harmonics of w_2 and some combinations $|w_1 - nw_2|$ ($n = 1, 2, 3$) (Fig. 2a). For $b > 1.029$ the high frequency peak and the harmonics are slowed down to noisy bands and a widening in the low frequency peak appears, as can be seen in Fig. 2b. For some particular values of b in this range, quasiperiodic windows emerge. In this situations, the spectrum is formed anew by sharp peaks.

To confirm the characteristics of the observed bifurcations, the larger Lyapunov exponent λ is studied. (This has been calculated by the Jacobian standard method on 10^3 points of a trajectory [Bergé *et al.*, 1984]). The result is given in Fig. 3. λ goes to zero when a bifurcation point is approached. It also gives a zero value for the interval of b that corresponds to the appearance and development of the limit cycles. For some values of b, $\lambda < 0$ and only some points on the limit cycles are visited by the iterations (quasiperiodicity). For the value $b = 1.025$, λ becomes positive in small intervals that alternate with quasiperiodic windows ($\lambda < 0$). But when $b = 1.029$, λ reaches a value that corresponds to the beginning of a chaotic band. From Fig. 3, one can also deduce the existence of quasiperiodic windows where $\lambda < 0$ in this range of b.

In order to determine the complexity of the attractor, we study its invariant measure. The last have been determined numerically by taking 10^7 iterates. Even for the limit cycles, the invariant measure is not uniform: the probability is larger near p_4 than on the

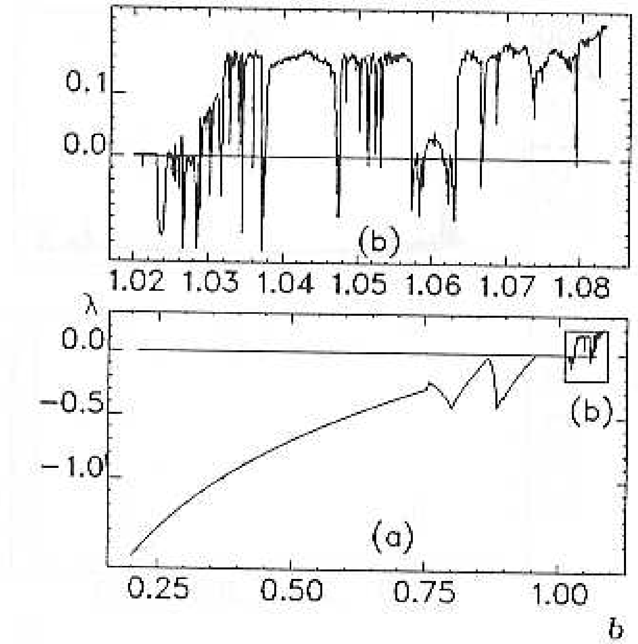

Fig. 3. (a) Largest Lyapunov exponent λ of the map as a function of the adjustable parameter b. (b) Enlarged view of $\lambda(b)$ in the chaotic region.

rest of the attractor. The attractor in the chaotic region ($1.029 < b < 1.08342$) increases its heterogeneity when b increases, but only in two very limited zones. The maximum of the invariant measure of the chaotic attractor [Eckmann & Ruelle, 1985] is localized in the neighborhood of p_4, while near the hyperbolic points $p_{1,2}$ the iterated points are scattered and therefore the invariant measure is minimal in this regions (see Fig. 4a). This can be studied quantitatively by determining the dimension spectrum $f(\alpha)$ as a function of the local dimensions α of the invariant measure [Halsey *et al.*, 1986]. The result given in Fig. 4b for $b = 1.0834$ has been calculated with the constant radius method taking 5000 points [Pawelzik & Schuster, 1987]. It confirms the multifractality of the attractor. The fractal dimension is $D_0 = 1.27$ and the local dimensions α reach values from 0.565 to 1.656.

3. Discussion and Conclusions

The coupling of two logistic maps has a deep effect on the solutions and the transition to chaos than that observed for the 1D logistic map [Yuan *et al.*, 1983]. This is a common conclusion in additive 2D maps with a diagonal symmetry [Yuan *et al.*, 1983; Hogg & Huberman, 1984]. We also confirm this conclusion when a multiplicative coupling is considered.

The bimap follows a route to chaos of the general form

fixed point — period-2 orbit — two limit cycles — chaos
frequencies: $w_1 = 1/2$ — w_1, w_2 — broad band

that corresponds qualitatively to the Ruelle–Takens route for 2D maps. In the case of two logistic maps with different additive couplings and with a diagonal symmetry, there exists a region in the space of parameters where the transition to chaos follows the same scheme [Yuan *et al.*, 1983].

However, the geometric mechanisms of transition is qualitatively different depending on the kind of coupling. In many cases [Yuan *et al.*, 1983; Chossat & Golubitsky, 1988] the chaotization is produced by local bifurcations in the limit cycle phase with the sequence:

two limit cycles — two chaotic bands — bands collision
$w_1 = 1/2, w_2 \neq 0$ — broad band around w_2 — broad band

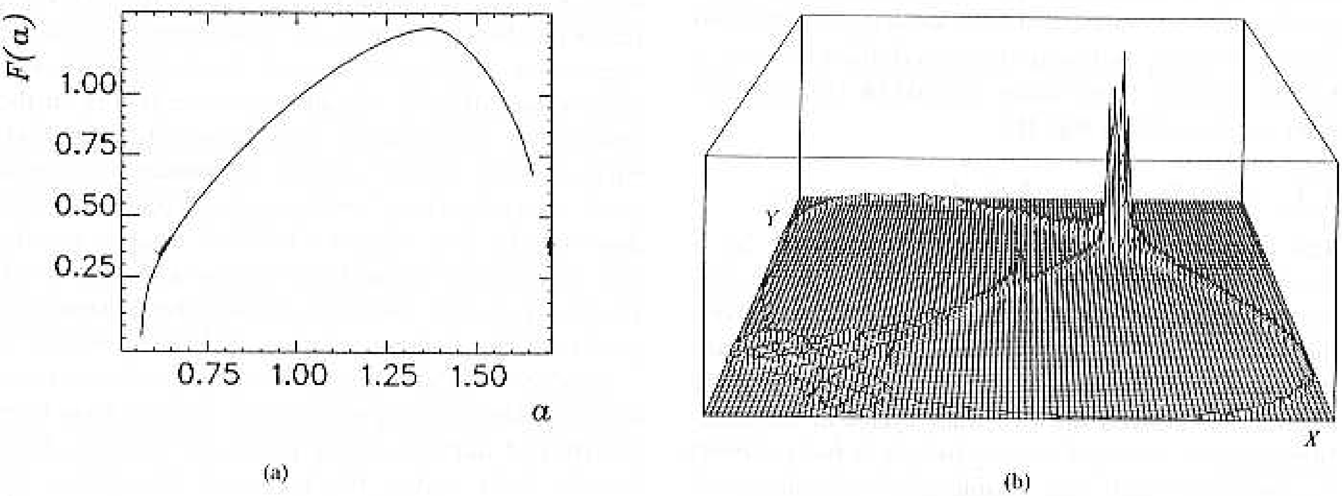

Fig. 4. (a) The $f(\alpha)$ spectrum of the map for $b = 1.0834$. (b) The invariant measure of the bimap computed from 10^6 iterates ($b = 1.0834$).

Instead, for the 2D map analyzed here, the two limit cycles remain until they suffer a global bifurcation due to the collision with the heteroclinic orbit on the diagonal, leading to chaos:

$$\text{two limits cycles} - \text{collision of the cycles with the diagonal}$$
$$w_1, w_2 - \text{broad band} \ .$$

This is qualitatively different from the scenario explained above. (Compare Fig. 2a in Yuan *et al.* [1983] and Fig. 1e in the present paper) The iterations in Fig. 1e. are distributed on two interlaced limit cycles, but chaos break the right–left diagonal alternancy (lose of $w_1 = 1/2$ frequency).

Moreover, we notice that this last scenario also can be obtained in the case of two coupled logistic maps with a diffusive coupling when the parameters are linked with special relationships [Kaneko, 1983; Jackson, 1990]. It seems that the symmetry of the map equations governs the sequence of bifurcations, and the kind of coupling does not seem to play a determinant role in this sequence. Further studies could allow the determination of the universality of this particular scheme within the general Ruelle–Takens route to chaos.

The attractor looks quite regular even in the chaotic region. However the analysis of the Lyapunov exponents confirms that chaos is taking place. The invariant measure and the dimension spectrum [$f(\alpha)$] confirms that the distribution of the iterates on the attractor is quite inhomogeneous. The maximum and the minima of the invariant measure of the chaotic attractor are concentrated in some very restricted zones near the unstable fixed points (Fig. 4a).

The present study suggests that more complex behavior will appear for higher dimensional coupling. In particular it would be interesting to analyze the different kinds of global couplings among a lattice of maps in order to search for universal properties.

Acknowledgements

We thank Dr. M. Bestehorn and Dr. G. Mindlin for helpful comments and discussions. This work was supported by a DGICYT (Spanish Ministry of Education) project PB90–0362. One of us (R. L-R) acknowledges the Gobierno Foral de Navarra (Spain) for a research grant.

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