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Citation: Chaos **28**, 075519 (2018); doi: 10.1063/1.5027615
View online: https://doi.org/10.1063/1.5027615
View Table of Contents: http://aip.scitation.org/toc/cha/28/7
Published by the American Institute of Physics

CHAOS 28, 075519 (2018)

Control and synchronization of hyperchaotic states in mathematical models of Bènard-Marangoni convective experiments

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(Received 4 March 2018; accepted 12 June 2018; published online 31 July 2018)

Mathematical models are of great interest for experimentalists since they provide a way for controlling and synchronizing different chaotic states. In previous works, we have used a Takens-Bogdanov (T-B) system under hyperchaotic dynamic conditions (two or more positive Lyapunov exponents) because they adequately reflect the dynamics of the patterns in small aspect ratio pre-turbulent Bènard-Marangoni convection near a codimension-2 point (with resonance between 2:1 modes), in square symmetry (D4). In this paper, we discuss the coupling of two different four dimensional hyperchaotic models derived from the Lorenz equations by using the same method introduced in previous works. As in the former system of used equations, we found that two identical hyperchaotic systems based on either Chen or Lü equation systems evolve into different states in the pattern space, where the synchronization state or the complexity could be controlled by a small external signal, as was shown in T-B equations. *Published by AIP Publishing.* https://doi.org/10.1063/1.5027615

Synchronization and control of complex dynamical states are very important tasks in fluid dynamics. Mathematical models *ad hoc* based on ordinary differential equations (ODE) that fit real experiments are very useful tools to help in the design of these systems. In previous works, we presented complete synchronization by coupling two equal hyperchaotic systems of Takens-Bogdanov (T-B) equations and their control by very small amplitude harmonic signals and tuning the frequency. In this work, we extend those results to two different systems of hyperchaotic equations derived from the Lorenz 3D model. Complete synchronization, as in the T-B system, and the results of control with a small amplitude signal tuning their frequency are shown.

I. INTRODUCTION

It is well known that when heat or mass is transported in a fluid by convective mechanisms, in the movements of the fluid can be found almost all the dynamic behaviors that appear in nature when a control parameter increases the transported magnitude. For instance, when a stationary fluid layer is heated from below, a flow of heat is transported from the bottom to the upper surface initially by conduction, as normally occurs in a solid. Increasing the applied temperature gradient sufficiently to overpass a threshold, the layer becomes unstable by buoyancy or surface tension and breaks the symmetry. The conductive stage is followed by convective movements in the fluid, forming self-ordered structures (or "patterns") in the temperature, density, pressure, and velocity fields.

These "patterns" are rolls, hexagons, with different forms like rolls, hexagonal, or could be also waves that appear near the convection threshold; depending on many parameters like boundary conditions properties and symmetries, and from thermal and mechanical properties of the fluid. The patterns could be stationary[1,2] or time-dependent.[3–6] Further increasing the control parameter, the structures become unstable producing movements that can be regular or chaotics, that is, time-dependent structures that finally give place to different pre-turbulent structures like vortices, thermal plumes, and bubbles moving in a main flow.[7–9] Finally, fully developed turbulent movements dominate the heat transport in the fluid, as can be observed every day in weather forecast in TV news. It is well known that Navier-Stokes partial differential equations (PDE) are the mathematical tools to model all the hydrodynamic descriptions in these kinds of problems in nature and in related experiments.[10] The difficulties to solve them with actual frames and real boundary conditions are also well known. Many approximations are necessary to solve these equations when nonlinearities become important or when the control parameter is far from threshold.

Since Lorenz published his model in 1963[11] to describe atmospheric convection incorporating in their solution flows "deterministic but not periodic" (that is, "chaotic dynamics"), many mathematical and numeric works have been performed on these non-linear dynamic equations and their chaotic behavior.[12] And new systems of equations appear to model low dimensional dynamics (3D) in many application fields such as reaction-diffusion mechanisms in chemistry, Lotka-Volterra in ecology, or Chua's circuits in electronics engineering.

Many of the equation systems used in applications, like Rössler equations,[13] are derived from the Lorenz model and consequently are related to the basic equations of hydrodynamics. More recently, other equations also related to the Lorenz model have been proposed like Refs. 14–16, which have been used in many fields of applications, such as thermoconvection experiments, and this is an additional reason that motivates our interest in these kind of mathematical models.

A few years after (1979), Rössler presented his equations for 3D systems. He modified their equation system to obtain a model involving regions of parameters with more than one positive Lyapunov exponent.[17] The result was a system with a dynamic that expands in more than one direction in the phase space producing more complex attractors. Rössler define as a "hyperchaotic system" to a chaotic system with more than one Lyapunov exponent positive. Consequently, hyperchaotic behavior arises as a natural regime in extended space time systems, in high dimensional confined oscillators, or in situations where many oscillators are coupled, as occurs, for example, in "Complex networks."[18]

Other systems of equations, with regions of parameters producing hyperchaos in confined oscillators, have been obtained by modifying an original 3D chaotic system of equations (that provide one positive Lyapunov exponent) by adding a coupled equation that transforms the system in 4D. Hyperchaos appears to adjust the parameters to obtain a region with two positive Lyapunov exponents in this 4D equation system. It is important to remark that even if having hyperchaotic oscillations is necessary to have at least four dimensions, the inverse is not true. Not all the 4D systems are hyperchaotic; they need to have at least two positive Lyapunov exponents.

The additional equation plays the role of feedback as in hyperchaotic Chen,[19] in a generalized Lorenz system,[20] or in hyperchaotic Lü systems.[21] Another important way to obtain hyper-chaos is to couple two (or more) chaotic systems avoiding the "chaos suppression" phenomena.[22]

Chaos suppression is a well-known effect related directly to this work and frequently considered as the opposite to "chaos generation." To suppress chaos, the positive Lyapunov exponents should be canceled. It has been obtained in the past by parameter perturbations or parametric forcing or modulation of the control parameter and can be considered among the methods to control chaos.[23,24]

A classical way to increase the order in a system is to couple two or more oscillators (similar or different) choosing carefully the parameters of coupling to obtain complete synchronization (CS). In 2002 was presented an example of chaos suppression through the coupling of two Duffing oscillators, one of them in a chaotic regime and the other in a periodic regime.[22]

In particular, chaos suppression by coupling two systems was demonstrated in 3D systems coupling asymmetrically two classical chaotic oscillators.[25] In this work, two identical Rössler (in funnel and no-funnel regimes), two Lorenz, and two Lotka-Volterra systems of equations have been coupled. The method used here with three 3D systems rests on selecting an adequate coupling in order to drive a chaotic dynamics towards a regular periodic attractor. As explained in this work, the problem fits in the more general frame of Synchronization[26] or in the general Theory of Oscillations.[27] In fact, numerical simulations display that for two coupled identical oscillators, the type and strength of coupling determine the Lyapunov exponents that can suppress chaos. This result in terms of synchronization theory means that the coupled system rests in a state of complete synchronization (CS), a kind of synchronization that in the new global system (composed now of the six equations coupled in 3D or eight equations in 4D) depends on the dynamics of the oscillators but also on the type of coupling between them, that is, number of links, symmetries, and strength of coupling. As an example, in Ref. 28, the synchronization that was obtained is directly related to the asymmetry of the coupling.

After these works, our interest was shifted to couple four dimensional hyperchaotic models. In this frame, first we presented the results obtained by coupling two T-B systems[28] and then other hyperchaotic systems with the aim of analyzing the influence of the type of coupling and strength coefficient on the type of chaos and synchronization that could be obtained, looking for the possibility of generic behaviors in 4D systems. Finally, we look for the possibility of controlling such complex systems previously coupled and synchronized. This can be done by studying the effects of harmonic signals with small amplitude injected to modify the state of the system as suggested in Ref. 29 but keeping their synchronization or dynamic possibilities as in Ref. 30. Following the latter work, in this paper, we use a method to choose the control frequencies based on histograms obtained from the signals observed. In fact, it is not possible to select the frequencies by classical Fourier methods as will be discussed in the next section (Sec. II).

As a resume, in this work, we extend the methods used in Ref. 30 to two hyperchaotic models ("Chen" and "Lü") that are related to the Lorenz system but with different topologies and dynamics, with the aim to check the possibility of the existence of generic behaviors in four dimensional systems. We present the results of self-synchronization by coupling two equal oscillators and the attempts to control the coupled systems by injecting small signals (less than 1% in amplitude of the variable x) and modifying their frequencies. A scheme of how systems are coupled and perturbed is shown in Fig. 1. Note that we are not changing the parameters, but we are just perturbing a coupled and synchronized system with a really small harmonic perturbation.

The work is organized as follows: after this Introduction follows Sec. II devoted to describe the dynamics of the used systems compared with the results obtained between both systems before coupling. Section III presents results of coupling on synchronization and chaos suppression. Section IV presents the results obtained in the driven systems when a signal is injected with three different frequencies. Finally, in Sec. V, the most important conclusions are summarized.

II. THE DYNAMICAL SYSTEMS

A. Chen and Lü hyperchaotic systems

In the light of the facts presented here, we decided to work with models associated with Lorenz generalized systems working in the hyperchaotic regime. The "Chen system," as it is usually named, was presented in Chen and Ueta[14] and in Celikovsky and Chen [15] as a dual system with respect to the Lorenz, in the sense defined by Celikovsky and Vanecek.[31] In fact, considering the linear part in the matrix of coefficients ($A = [a_{ij} 3 \times 3]$) of both systems, the Lorenz system satisfies that $a_{12}a_{21} > 0$, while for the Chen system $a_{12}a_{21} < 0$. This means that the systems have a different canonical family,[15]

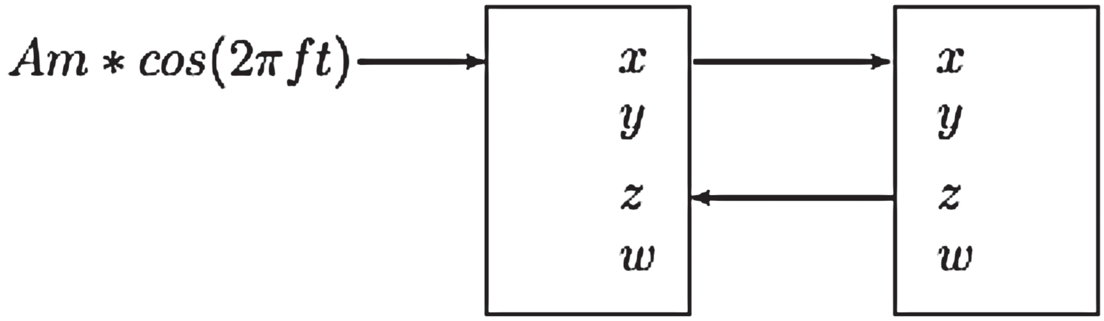

FIG. 1. Scheme of how dynamical systems are coupled and perturbed by a harmonic signal at frequency f.

and in spite of many similitudes they are topologically different. In 2002, Lü and Chen[16] completed the frame of the "generalized Lorenz systems" with another related chaotic system where $a_{12}a_{21} = 0$, just in the transition between Lorenz and Chen systems. This system is usually named "Lü."

A mathematical proof for the Chen attractor existence based on the fundamental works of Shil'nikov can be found in Ref. 32. Authors display the existence of heteroclinic and homoclinic orbits using the Shil'nikov criteria,[33] verifying that the Chen system has both kinds of Smale's horseshoe type of chaos. In spite of these and other important analytic results, the work made to know the behavior of these systems against parameters has been mostly numeric.

Chen chaotic three dimensional system (3D) has been transformed later (2006) into hyperchaotic (4D), simply by adding a linear controller to the second equation of a 3D Chen chaotic system.[19,20] By a similar method, hyperchaotic states in a 3D Lü system have been generated introducing a state feedback controller, transforming the system into 4D. These numerical results have been verified experimentally in

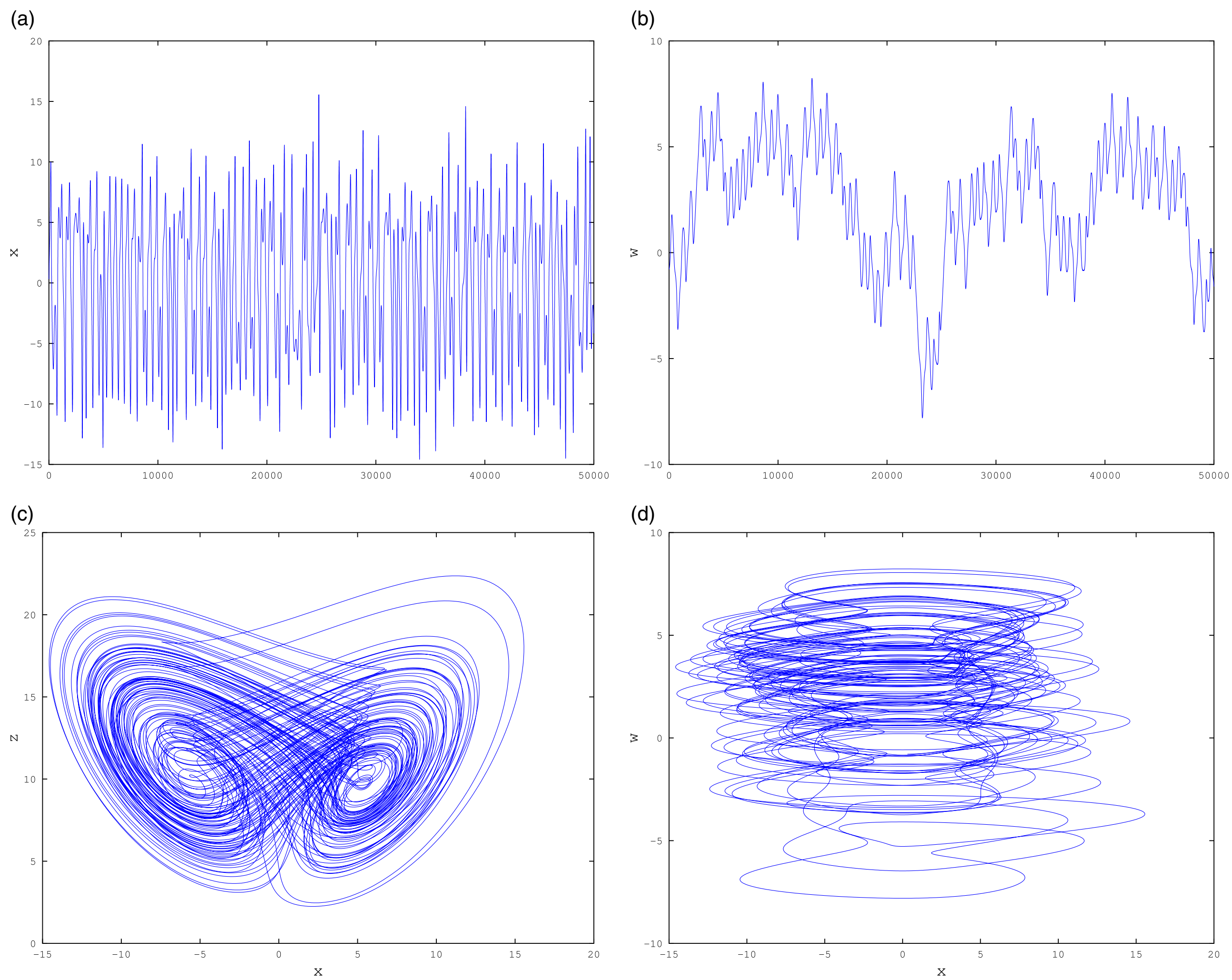

FIG. 2. Time series and phase space for Chen System under different variables and planes. (a) Time series for $x(t)$ variable. (b) Time series for $w(t)$ variable. (c) Phase space for the plane (x–z). (d) Phase space for the plane (x–w).

electronic circuits.[21] In this work, we are using the following equation systems:

$$\text{hyperchaotic Chen system}\begin{cases}\dot{x} &= a(y-x)\\ \dot{y} &= -dx - xz + cy\\ \dot{z} &= xy - bz\\ \dot{w} &= x + k\end{cases}, \quad (1)$$

where $a = 36, b = 3, c = 26, d = 16$, and $k = 0$ and

$$\text{hyperchaotic Lü system}\begin{cases}\dot{x} &= a(y-w)\\ \dot{y} &= -x - xz + cy\\ \dot{z} &= xy - bz\\ \dot{w} &= xy + dw\end{cases}, \quad (2)$$

where $a = 36, b = 3, c = 20$, and $d = 1$.

It has been mentioned that a hyperchaotic state requires at least two positive Lyapunov exponents. In a continuous four dimensional dissipative system like the one we are considering here, there are four possible types of attractors classified by their Lyapunov exponents $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Hyperchaotic attractor: $(\lambda_1, \lambda_2 > 0, \lambda_3 < 0, \lambda_4 = 0)$.

Chaotic attractor: $(\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0, \lambda_4 = 0)$.

Periodic orbits: $(\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0, \lambda_4 = 0)$.

Quasi-periodic orbits: $(\lambda_1 < 0, \lambda_2 < 0, \lambda_3 = 0, \lambda_4 = 0)$.

Equilibrium point: $(\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0, \lambda_4 < 0)$.

A Chen system in the hyperchaotic regime produces temporal files of variables like $x(t)$ and $w(t)$, which are depicted in Figs. 2(a) and 2(b). The phase space of the Chen attractor can be observed by projecting on the planes $(z–x)$ and $(w–x)$ [see Figs. 2(c) and 2(d)]. We have selected the planes of phase projections appearing in the figures to understand better the effects of coupling two systems and the effects of the injected control signals. The corresponding figures for temporal variables appearing in the Lü hyperchaotic attractor are depicted in Figs. 3(a) and 3(b) for $x(t)$ and $w(t)$, respectively. Moreover, their space phase for the two planes (x, z) and (x, w) is illustrated in Figs. 3(c) and 3(d), respectively.

Additional details as dependence of Lyapunov exponents with parameters or bifurcation diagrams can be found in Ref. 19. An important remark is to note that in Ref. 34 has been shown by analytical and numerical methods the riddled property for the Chen attractor, a fact that gives to the attractor some unusual properties. This property is shared with the T-B system studied in Ref. 28.

III. COUPLING EFFECTS

It is well known that complexity in the global dynamics of coupled systems depends on the dynamics of each oscillator

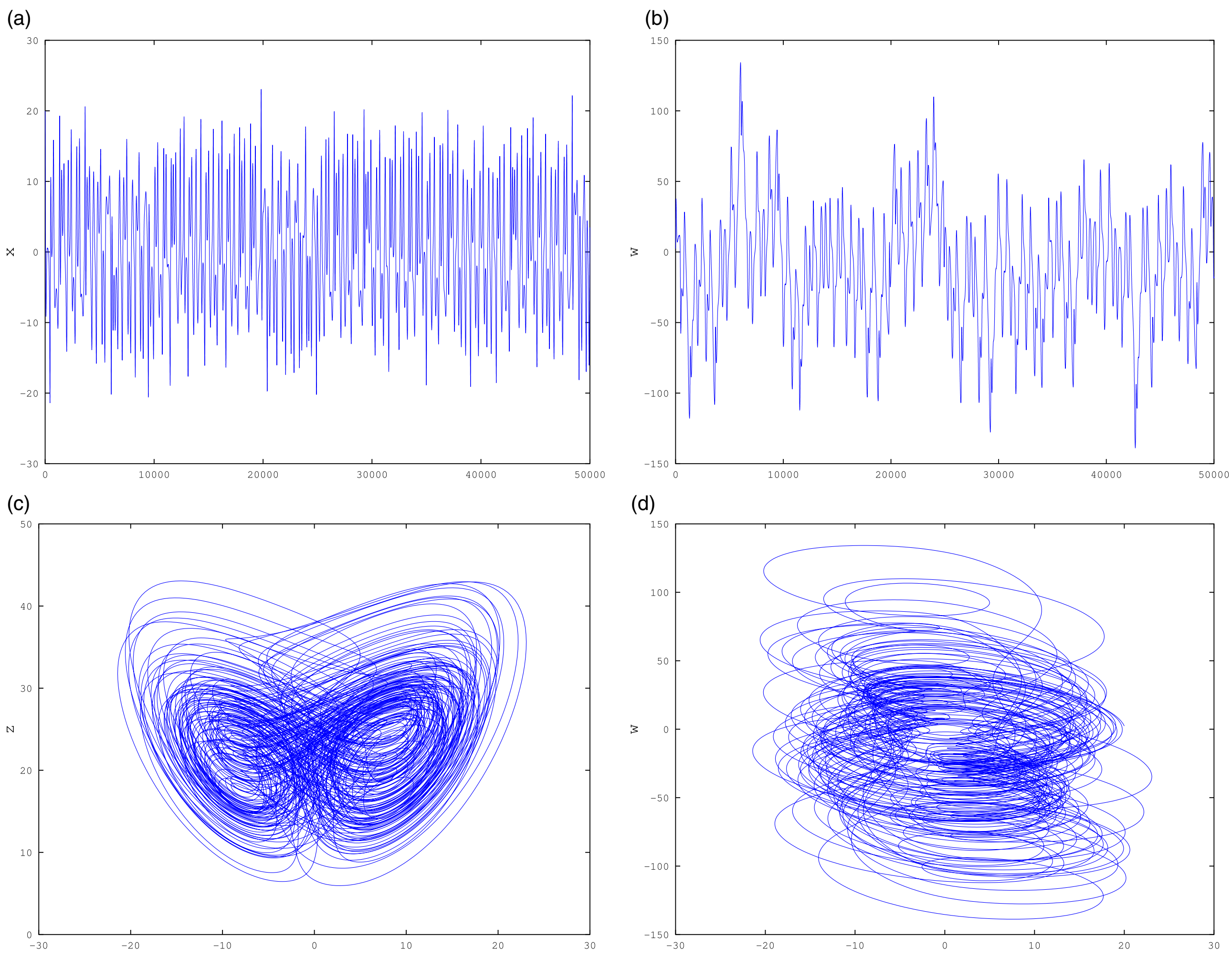

FIG. 3. Times series and phase space for Lü System under different variables and planes. (a) Time series for $x(t)$ variable. (b) Time series for $w(t)$ variable. (c) Phase space for the plane $(x–z)$. (d) Phase space for the plane $(x–w)$.

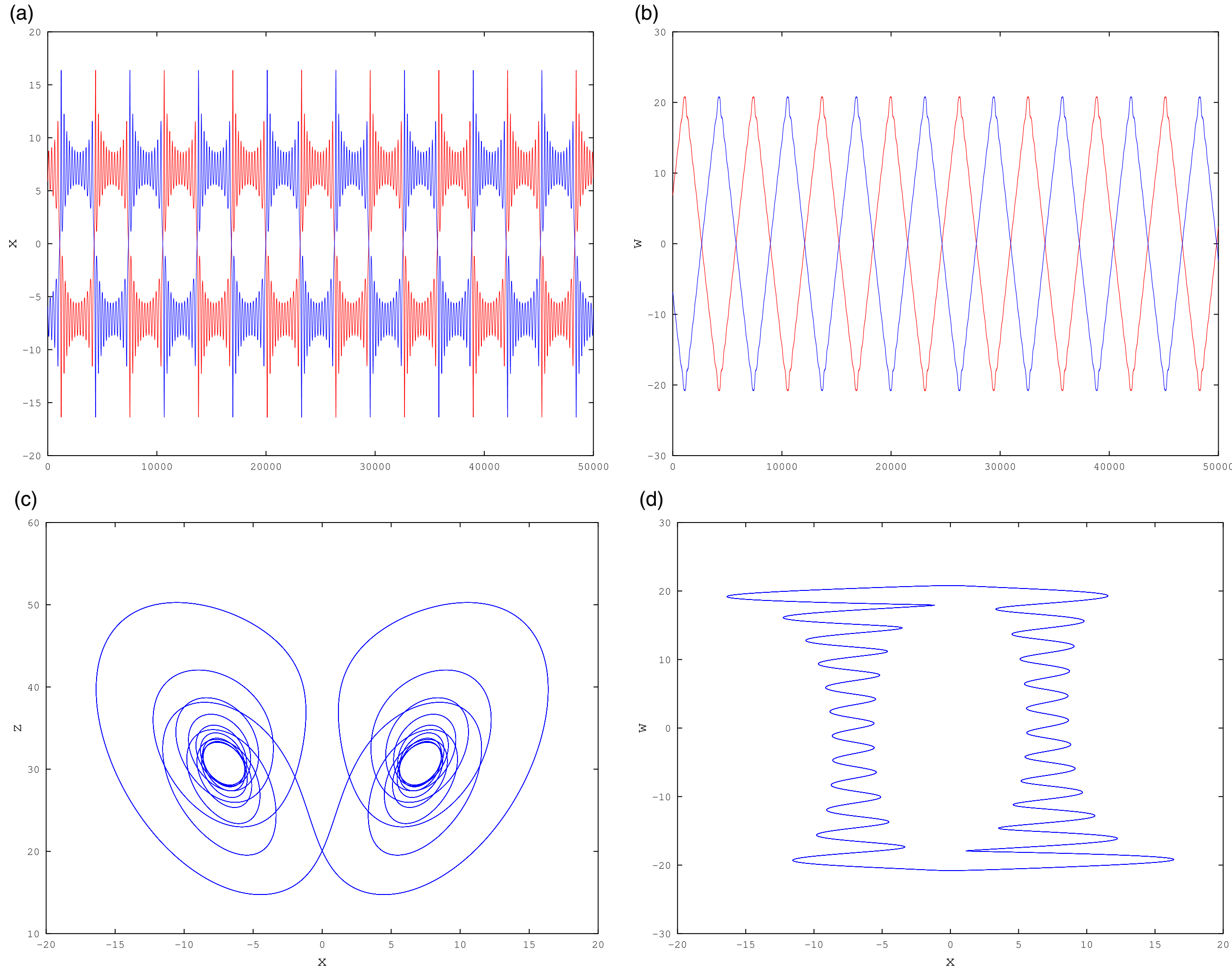

FIG. 4. Time series and phase space under different planes and variables for Chen coupled system. (a) Time series for $x(t)$ variable for systems 1 and 2: blue for $x_1(t)$, whereas red for $x_2(t)$. (b) Time series for $w(t)$ variable for systems 1 and 2: blue for $w_1(t)$, whereas red for $w_2(t)$. (c) Phase space for the projection (x–z). (d) Phase space for the projection (x–w).

and also on symmetries and strength of the external links. The links transform two isolated dynamics (here four dimensional systems) into a new system of eight equations, a new system that will have a collective regime with an emerging behavior that will not only depend on the dynamics that each oscillator is able to generate but also on the effects produced by the type and strength of the coupling between variables of the system. It is important to recall that in four dimensional oscillators, equivalent variables could be coupled in many different ways going from one to four links coupling all the equations that additionally could be in one or two directions (asymmetric or symmetric).

Synchronization induced by asymmetric coupling (master-slave configuration) of two or more autonomous oscillators is a well-known effect.[25,28] In four dimensional systems (4D) with symmetrical coupling of two identical T-B systems, one variable (x) has been presented in Ref. 28.

Synchronization regimes of these and other equations (principally 3D) analyzed have also used different coupling schemes (symmetric or asymmetric), considering the coupling as a direct function of the error between both systems acting as a feedback loop. Phase synchronization (PS) has been obtained using selected values for parameters that must be fine-tuned to fit Lyapunov exponent windows.[25] The synchronized regimes obtained in this case are not too stable and depend strongly on the coupling coefficient.

Complete synchronization (CS) as a result of coupling in four dimensional oscillators was observed in the hyperchaotic T-B system.[28] In this work, asymmetric coupling was chosen to obtain a robust synchronization manifold recovering for the coupled system the internal symmetry of each oscillator by coupling the feedback on two different variables (x and z). Internal symmetry of the equations of the global system couples the variables by pairs (x and y to z and w).

Introducing the coupling between oscillators 1 and 2 in this way appears as a closed "ring structure" that provides the robustness necessary to obtain complete synchronization (CS), a fact that was impossible to obtain in the case of the symmetrical coupling scheme based on only one variable. This situation was explored by studying the dependence of Lyapunov exponents against the two parameters of the coupling strength $\varepsilon_x(x_2 - x_1)$ and $\varepsilon_z(z_1 - z_2)$.

In other previous works before coupling the oscillators that are analyzed here, we have verified that each one of

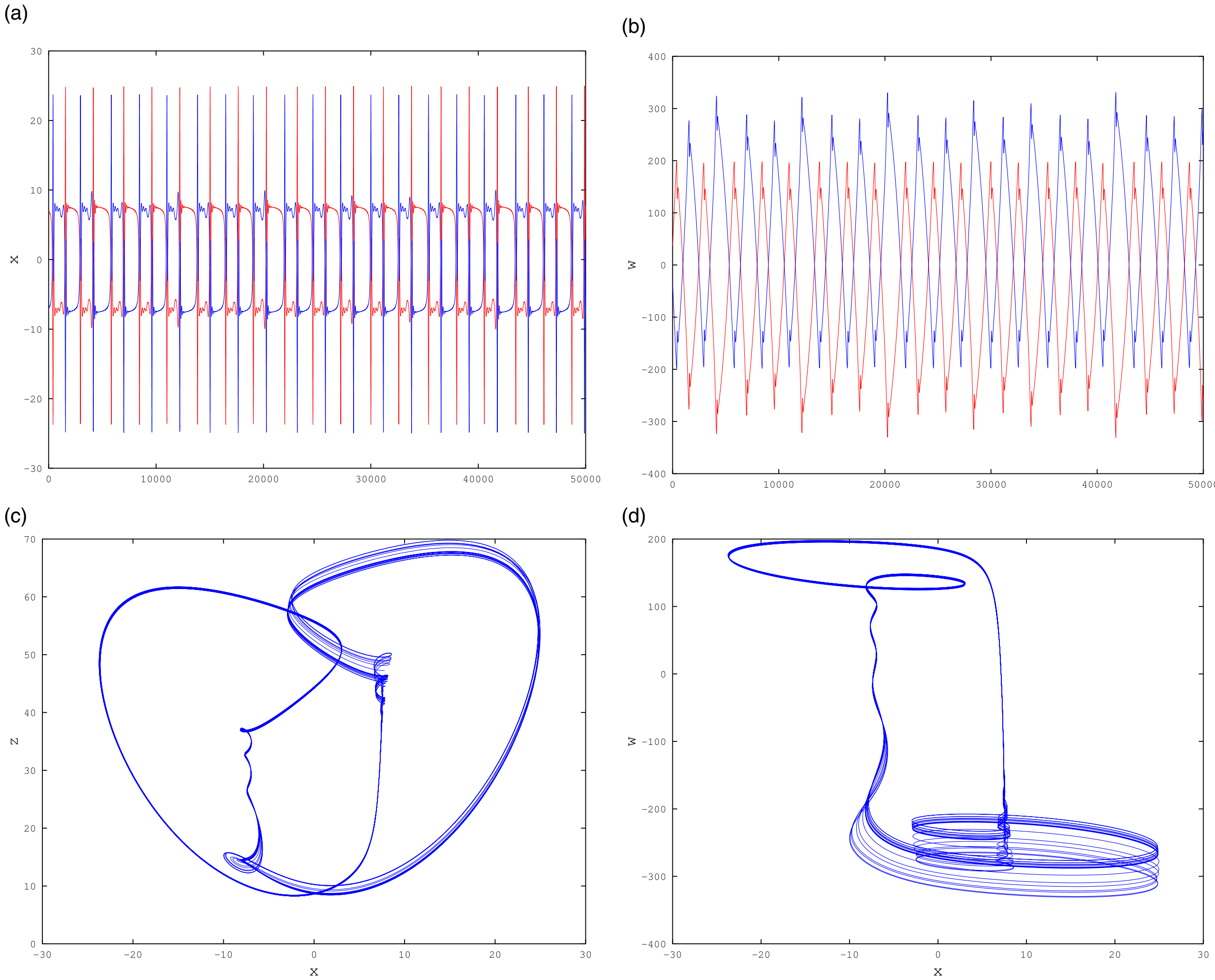

FIG. 5. Time series and phase space under different planes and variables for Lü coupled system. (a) Time series for $x(t)$ variable for systems 1 and 2: blue for $x_1(t)$, whereas red for $x_2(t)$. (b) Time series for $w(t)$ variable for systems 1 and 2: blue for $w_1(t)$, whereas red for $w_2(t)$. (c) Phase space for the projection (x–z). (d) Phase space for the projection (x–z).

them is able to display all the dynamical states like stationary solutions, oscillatory with simple or multiple periods, chaotic solutions, and more complex hyperchaotic situations as a function of the parameters. When the possibility for synchronization is determined by looking at the behavior of the largest Lyapunov positive exponents (or the sum of all of them), it is important to detect in the parameter space the values for which synchronization is possible and to know the

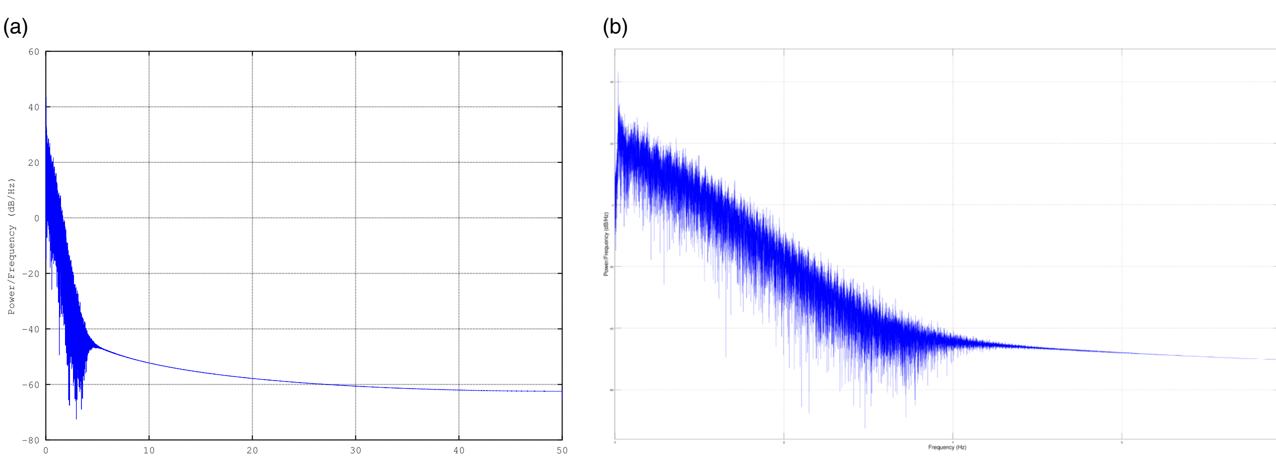

FIG. 6. Lü system Fourier spectrum. (a) $x(t)$ Fourier spectrum. (b) Zoomed-in $x(t)$ Fourier spectrum.

values of the strength of coupling that are necessary. Frequently appears this possibility only for narrow windows in the parameter space.

A classification of the synchronization states of these attractors used in this work can be found in Refs. 35 and 28. We consider here these results as our reference in 4D systems because it is possible to follow the sequence of results from the experiment in hydrodynamics[3] to the attempts to synchronize two equal mathematical systems and the control of complexity by harmonic signals of small amplitude.[30]

A. The coupled equations

The coupled system transforms the set of four equations (1) into other higher dimensional systems (eight equations), where variables are named by subscripts 1 and 2, corresponding to each original system. Each oscillator uses the adjusted parameter with the aim to be in a hyperchaotic region (with more than one positive Lyapunov exponent).

The coupling terms ε_x and ε_z can be interpreted as the feedback for systems reproducing the same trajectory on the synchronized manifold separately without any feedback between them. This kind of coupling extends the inner symmetries of each equation system to the coupled one which has a higher dimension.

Regarding the results shown, it is important to remark that the simulations have a transitory stage where systems are iterated 10^6 time units. On the second phase, the systems are iterated 10^6 time units to obtain the output signal. For the sake of clarity, we plot just some chunk of these signals.

$$
\begin{aligned}
\dot{x}_1 &= a(y_1 - x_1) + \varepsilon_x(x_2 - x_1),\\
\dot{y}_1 &= -dx_1 - x_1 z_{1,2} + cy_1 - w_1,\\
\dot{z}_1 &= x_1 y_1 - bz_1 + \varepsilon_z(z_2 - z_1),\\
\dot{w}_1 &= x_{1,2} + k,\\
\dot{x}_2 &= a(y_2 - x_2) + \varepsilon_x(x_1 - x_2),\\
\dot{y}_2 &= -dx_2 - x_2 z_2 + cy_2 - w_2,\\
\dot{z}_2 &= x_2 y_2 - bz_2 + \varepsilon_z(z_1 - z_2),\\
\dot{w}_2 &= x_2 + k,
\end{aligned} \tag{3}
$$

where $a = 36$, $b = 3$, $c = 26$, $k = 0$, and $\epsilon_x = 14.5$ and $\epsilon_z = 14.5$.

$$
\begin{aligned}
\dot{x}_1 &= a(y_1 - w_1) + \varepsilon_x(x_2 - x_1),\\
\dot{y}_1 &= -x_{1,2} - x_{1,2} z_{1,2} + cy_{1,2},\\
\dot{z}_1 &= x_{1,2} y_{1,2} - bz_1 + \varepsilon_z(z_2 - z_1),\\
\dot{w}_1 &= x_1 y_{1,2} + dw_1,\\
\dot{x}_2 &= a(y_{2,1} - w_2) + \varepsilon_x(x_1 - x_2),\\
\dot{y}_2 &= -x_2 - x_2 z_2 + cy_2,\\
\dot{z}_2 &= x_2 y_2 - bz_2 + \varepsilon_z(z_1 - z_2),\\
\dot{w}_2 &= x_2 y_2 + dw_2,
\end{aligned} \tag{4}
$$

where $a = 36$, $b = 3$, $c = 20$, and $d = 1$ and the coupling factors are defined as $\varepsilon_x = 10$ and $\varepsilon_z = 10$.

The results obtained for the Chen coupled system are depicted in Fig. 4. Figures 4(a) and 4(b) display that x_2 is completely synchronized to x_1 and w_2 completely synchronized to w_1. To represent the attractor, we choose the projection onto the planes (z–x), where the result is illustrated in Fig. 4(c), whereas for the projection (w–x) the corresponding graph is depicted in Fig. 4(d).

Similar results obtained in the Lü system are illustrated in Fig. 5 with identical distribution of variables. Comparing these results with those in Figs. 2 and 3, we observe in temporal signals that complete synchronization (CS) has been obtained with some variables like x or w being synchronized in counter phase and others in phase.

IV. DRIVING THE COUPLED SYSTEMS WITH HARMONIC SIGNALS

The used hyperchaotic oscillators have some general features in their dynamics that should be taken into consideration. One of these general results is the common picture of the

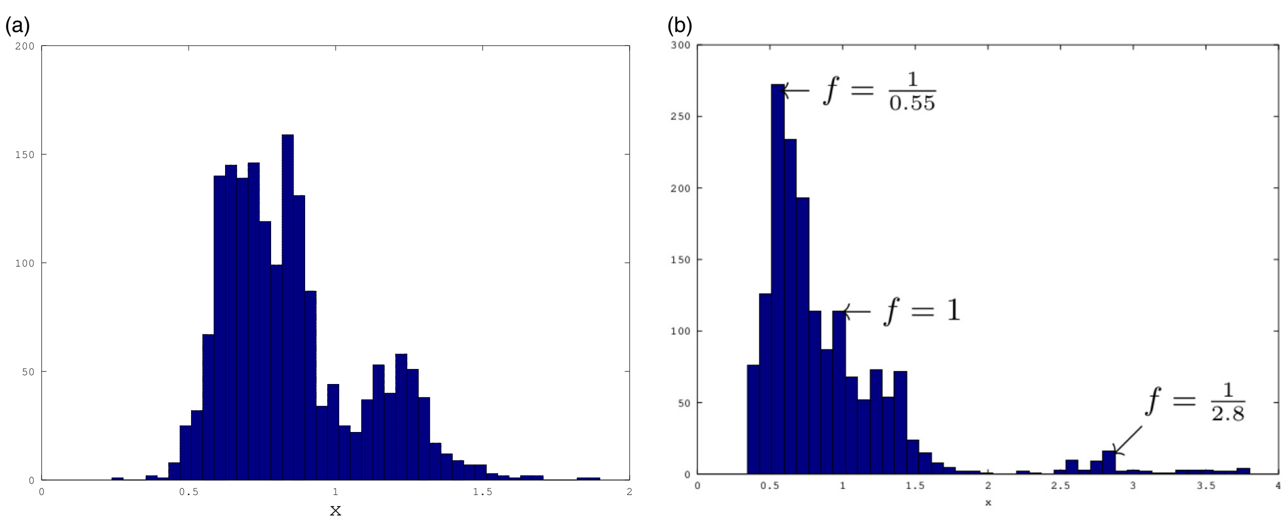

FIG. 7. Lü system histogram. (a) $x(t)$ histogram for uncoupled system. (b) $x(t)$ histogram for coupled system.

Fourier spectrum of the main variables obtained with long temporal data files.

We have checked that the spectrum of these hyperchaotic systems has normally a low frequency region over-imposed to the characteristic chaotic $1/f$ noise, followed by a long tail, as shown in Fig. 6(a). The existence of this common shape in the Fourier spectrum was observed in Takens-Bodanov[28,30] and in Chen and Lü hyperchaotic systems.[35] The Fourier

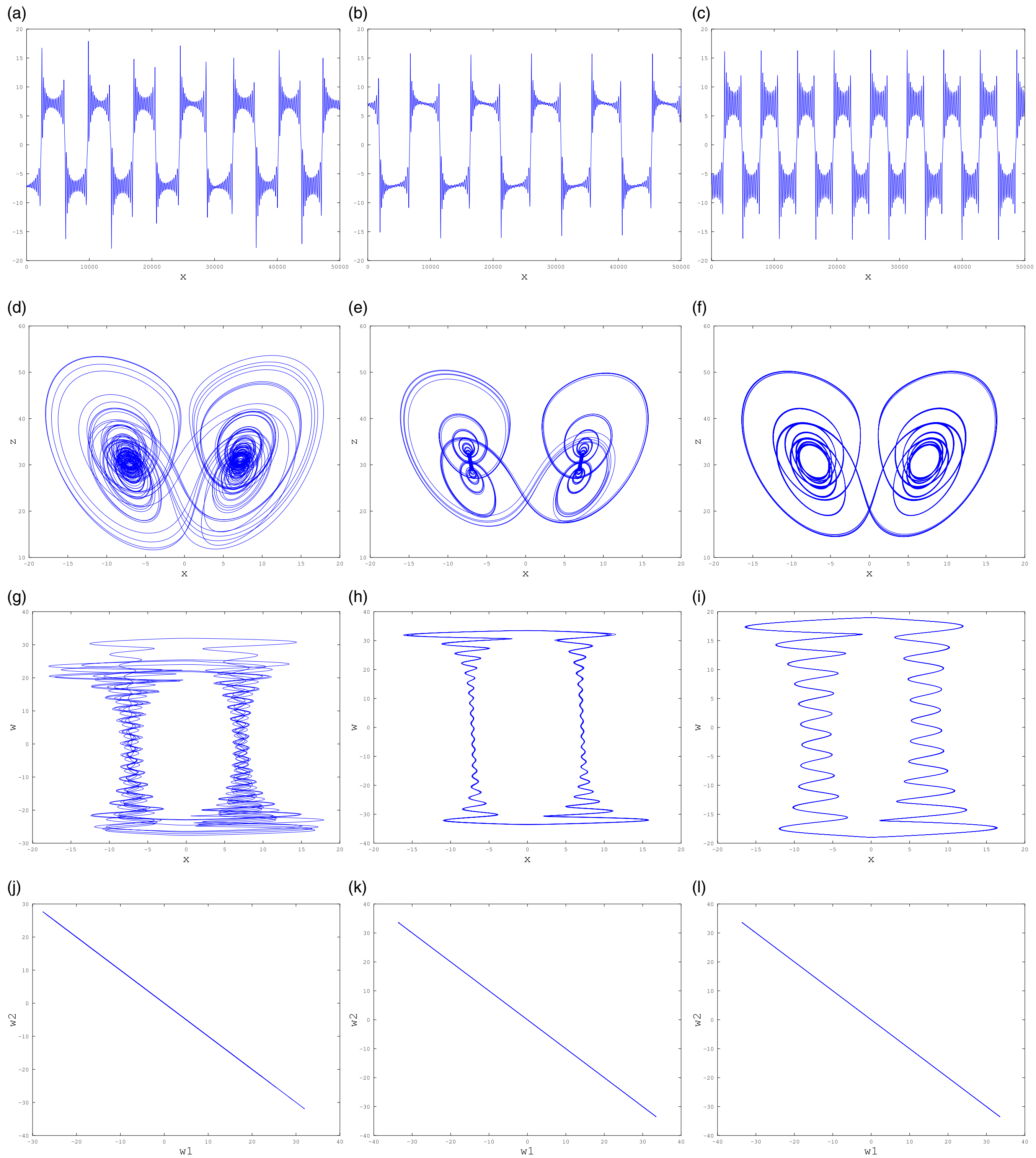

FIG. 8. Impact of the selected frequency for the injected function on the behavior of Chen coupled system. (a) Time series for $x(t)$ by using an injection function under $f = 1/0.49$. (b) Time series for $x(t)$ by using an injection function under $f = 1/0.78$. (c) Time series for $x(t)$ by using an injection function under $f = 1/0.9$. (d) Phase space for the projection (x–z) by using an injection function under $f = 1/0.49$. (e) Phase space for the projection (x–z) by using an injection function under $f = 1/0.78$. (f) Phase space for the projection (x–z) by using an injection function under $f = 1/0.9$. (g) Phase space for the projection (x–w) by using an injection function under $f = 1/0.49$. (h) Phase space for the projection (x–w) by using an injection function under $f = 1/0.78$. (i) Phase space for the projection (x–w) by using an injection function under $f = 1/0.9$. (j) $x_1(t)$ versus $x_2(t)$ using an injection function under $f = 1/0.49$. (k) $x_1(t)$ versus $x_2(t)$ using an injection function under $f = 1/0.78$. (l) $x_1(t)$ versus $x_2(t)$ using an injection function under $f = 1/0.9$.

spectrum displayed in Fig. 6(a) for the variable x is enlarged to see a detail of the low frequency region for the Lü system [see Fig. 6(b)]. It is easy to see that not too much information could be obtained from this Fourier spectrum in order to choose a frequency for synchronization or control the system.

As in many other systems (like the Rössler attractor in the "funnel" parameter conditions[13]), it is not possible to construct a Poincaré section, and under this restriction, we cannot define an "analytical phase"[26] that helps us to synchronize or control the system.

To overcome this problem, we calculate a histogram of a data file like $x(t)$ to see the most relevant characteristic periods that appear in the system as described in Ref. 30. Periods are obtained from a long signal data file measuring the time between two maximum or minimum first neighbors in the signal. The histogram represents the frequencies (in number on times) that each period appears in a long signal data file. A typical histogram appears in Fig. 7(b).

By using the information obtained in this recurrence time distribution, it is possible to analyze better the effects of a noiseless harmonic signal injected to the synchronized system [as shown in Fig. 7(a)]. This histogram is not a Poincaré section and times are not the classic "return times," but for a long file which represents the more frequently visited periods in the system, are connected with the homoclinic and heteroclinic orbits.

In Figs. 8 and 9, the effects of the signals injected with the frequencies chosen to fit some of the peaks observed in the time recurrence plot are compared with the effects of other two signals with other frequencies chosen in other two regions of the histogram. In all cases, the amplitude will be the same, a small fraction (0.1% of the variable maximum value). The results are shown in Fig. 8 for the Chen system and in Fig. 9

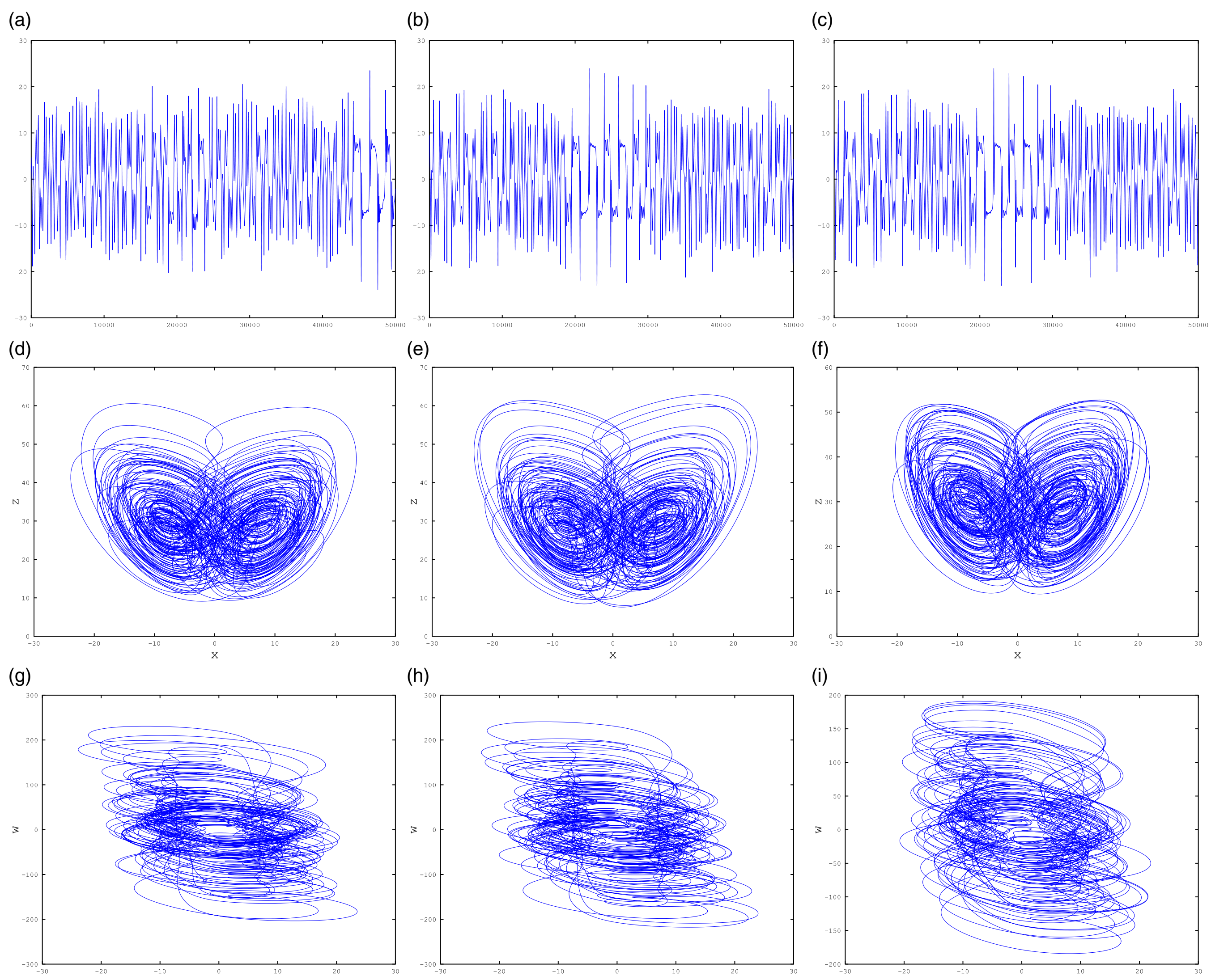

FIG. 9. Impact of the selected frequency for the injected function on the behavior of Lü coupled system. (a) Time series for $x(t)$ by using an injection function under $f = 1/0.55$. (b) Time series for $x(t)$ by using an injection function under $f = 1$. (c) Time series for $x(t)$ by using an injection function under $f = 1/2.8$. (d) Phase space for the projection (x–z) by using an injection function under $f = 1/0.55$. (e) Phase space for the projection (x–z) by using an injection function under $f = 1$. (f) Phase space for the projection (x–z) by using an injection function under $f = 1/2.8$. (g) Phase space for the projection (x–w) by using an injection function under $f = 1/0.55$. (h) Phase space for the projection (x–w) by using an injection function under $f = 1$. (i) Phase space for the projection (x–z) by using an injection function under $f = 1/2.8$.

for the Lü system. Both figures have been ordered in a column with the results for each frequency injected. The results corresponding to the variable $x(t)$ is depicted in the first row of the file; phase space projection (z–x) is illustrated in the second row of file and (w–x) projection is depicted in the third row for each frequency for three different signals with different frequencies.

For the Chen system, there is an additional row file to display the synchronization between the signals that remain synchronized (CS) in spite of the different frequencies of the injected signals applied. The results display a tuning to three different states (keeping the system synchronized).

The Lü system loses the complete synchronization, but the signal moves the equilibrium to a region of the attractor with a tangent bifurcation having an intermittent behavior with a laminar region that can be seen clearly in the central column for the second frequency injected (Fig. 9). The system could keep general synchronization but with intermittent chaotic bursts.

V. CONCLUSIONS

In this work, we display many new details about the behavior of Chen and Lü systems in the hyperchaotic regime obtained by numerical simulation that are summarized in Figs. 2 and 3, respectively.

Two identical hyperchaotic oscillators with equation systems Chen and Lü have been coupled as in the case of T-B, to verify whether they become synchronized when they are symmetrically coupled with the appropriate strength of coupling. This important result has been checked by using symmetrical coupling first with one link in variable x, and then with two links (in x and z). Complete synchronization has been obtained in both cases by coupling, but robustness improves by using two links. The results for two links are displayed in Figs. 4 and 7, and to our knowledge, this is the first report of complete synchronization with these systems of equations and types of links.

After the coupled systems are synchronized, a harmonic signal of very small amplitude (i.e., 0.1% of x amplitude) was injected to each system to check if a control of the complexity of the system is possible, as occurs in the T-B system.[30] The results in the Chen system, which is depicted in Fig. 8, generalize to this system the possibility of controlling complexity with small amplitude signals by simply tuning the frequency.

Figure 9 displays for the Lü system similar results showing an intermittent burst of laminarity in a hyperchaotic signal, a fact that informs us about the existence of a tangent bifurcation in the attractor. It appears as relevant that, in spite of the hyperchaotic character of these systems, some control can be obtained with very small amplitude signals by only adjusting the frequencies to specific values.

To control these systems of equations as in our former work with T-B equations, we use histograms constructed *ad hoc*. These histograms represent the most characteristic times, or the more frequently visited times in the data sequence, and confirm that the method introduced in Ref. 30 is more useful than the Fourier spectrum to choose the frequencies of the injection signal. Additional work is necessary to obtain a close relationship between the histograms and the heteroclinic and homoclinic orbits in the phase space.[36,37]

ACKNOWLEDGMENTS

We are grateful for financial support from: Education and Science Ministry (MEC) projects FIS2014-54101-P and FIS2017-83401-P, Spain. R.B. and G.V. have been supported by Enigmedia. H.M. acknowledges J. Burguete University of Navarra (UNAV), Spain for helpful comments.

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